

The CPT and Bisognano-Wichmann Theorems for Anyons and Plektons in $d=2+1$

Jens Mund*

Departamento de Física, Universidade Federal de Juiz de Fora,
36036-900 Juiz de Fora, MG, Brazil.
E-mail: mund@fisica.ufjf.br

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Dedicated to the memory of Bernd Kuckert.

Abstract

We prove the Bisognano-Wichmann and CPT theorems for massive particles obeying braid group statistics in three-dimensional Minkowski space. We start from first principles of local relativistic quantum theory, assuming Poincaré covariance and asymptotic completeness. The particle masses must be isolated points in the mass spectra of the corresponding charged sectors, and may only be finitely degenerate.

Introduction

The Bisognano-Wichmann theorem states that a large class of models in relativistic quantum field theory satisfies modular covariance, namely: The modular unitary group [8] of the field algebra associated to a (Rindler) wedge region coincides with the unitary group representing the boosts which preserve the wedge. Since the boosts associated to all wedge regions generate the Poincaré group, modular covariance implies that the representation of the Poincaré group is encoded intrinsically in the field algebra. This has important consequences, most prominently the spin-statistics theorem, the particle/anti-particle symmetry and the CPT theorem [29,34]. Modular covariance also implies a maximality condition for the field algebra, namely the duality property [2], and it implies the Unruh effect [57], namely that for a uniformly accelerated observer the vacuum looks like a heat bath whose temperature is (acceleration)/ 2π . The original theorem of Bisognano and Wichmann [2,3] relied on the CPT theorem [45] and was valid for finite component Wightman fields. However, the physical significance of this latter hypothesis is unclear. In the framework of algebraic quantum field theory [1,31], Guido and Longo have derived modular covariance in complete generality for conformally covariant theories [9]. In the four-dimensional Poincaré covariant case the Bisognano-Wichmann theorem has been shown by the author [39] to hold under physically transparent conditions, namely for massive theories with asymptotic completeness. (Conditions of more technical nature have been found by several authors [5,7,30,35,54], see [7] for a review of these results.) In three-dimensional spacetime, however, there

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may be charged sectors with braid group statistics [19, 24] containing particles whose spin is neither integer nor half-integer, which are called Plektons [23] or, if the statistics is described by an Abelian representation of the braid group, Anyons [59]. In this case, modular covariance and the CPT theorem are also expected [25, Assumption 4.1] to hold under certain conditions, but so far have not been proved in a model-independent way. The aim of the present article is to prove these theorems from first principles for massive Poincaré covariant Plektons satisfying asymptotic completeness.

Let us comment in more detail on the Bisognano-Wichmann and CPT theorems, and their interrelation, in the familiar case of permutation group statistics. Let W_1 be the standard wedge

$$W_1 := \{x \in \mathbb{R}^3 : |x^0| < x^1\}. \quad (1)$$

The Tomita operator associated with the field algebra of W_1 and the vacuum is defined as the closed anti-linear operator S satisfying

$$S F \Omega := F^* \Omega, \quad F \in \mathcal{F}(W_1), \quad (2)$$

where F^* is the operator adjoint and $\mathcal{F}(W_1)$ denotes the algebra of fields localized in W_1 . Denoting its polar decomposition by $S = J \Delta^{1/2}$, J and Δ^{it} are called the modular conjugation and modular unitary group, respectively, associated with $\mathcal{F}(W_1)$ and Ω . *Modular covariance* means that the modular unitary group coincides with the unitary group representing the boosts in 1-direction (which preserve the wedge W_1), namely:

$$\Delta^{it} = U(\lambda_1(-2\pi t)), \quad (3)$$

where $\lambda_1(t)$ acts as $\cosh(t) \mathbb{1} + \sinh(t) \sigma_1$ on the coordinates x^0, x^1 . Here, U is the representation of the universal covering group, \tilde{P}_+^\dagger , of the Poincaré group under which the fields are covariant. (Note that then, by covariance, the modular groups associated to other wedges $W = gW_1$ represent in the same way the corresponding boosts $\lambda_W(t) = g\lambda_1(t)g^{-1}$, and hence the entire representation U is fixed by the modular data.) The CPT theorem, on the other hand, asserts the existence of an anti-unitary CPT operator Θ which represents the reflexion¹ $j := \text{diag}(-1, -1, 1)$ at the edge of the standard wedge W_1 in a geometrically correct way:²

$$\Theta U(\tilde{g}) \Theta^{-1} = U(j\tilde{g}j), \quad \tilde{g} \in \tilde{P}_+^\dagger, \quad (4)$$

$$\text{Ad}\Theta : \mathcal{F}(C) \rightarrow \mathcal{F}(jC). \quad (5)$$

Here, C is a spacetime region within a suitable class. Further, if a field F carries a certain charge then

$$\Theta F \Theta^{-1} \text{ carries the conjugate charge.} \quad (6)$$

The CPT theorem has been used as an input to the proof of modular covariance by Bisognano and Wichmann [2, 3]. Conversely, the work of Guido and Longo [29], and Kuckert [34], has shown that modular covariance implies the CPT theorem. In particular, Guido and Longo have shown [29] that modular covariance of the *observable* algebra $\mathcal{A}(W_1)$ implies that the corresponding modular conjugation is a “PT” operator on the

¹We consider j as the *PT* transformation. The total spacetime inversion arises in four-dimensional spacetime from j through a π -rotation about the 1-axis, and is thus also a symmetry. In the odd-dimensional case at hand, j is the proper candidate for a symmetry (in combination with charge conjugation), while the total spacetime inversion is not – in fact, the latter *cannot* be a symmetry in the presence of braid group statistics [25].

²In Eq. (4), $j\tilde{g}j$ denotes the unique lift [58] of the adjoint action of j from P_+^\dagger to \tilde{P}_+^\dagger .

observable level, namely satisfies Eq. (4) on the vacuum Hilbert space and Eq. (5) with $\mathcal{A}(C)$ instead of $\mathcal{F}(C)$. Further, it intertwines a charged sector with its conjugate sector in the sense of representations, see Eq.s (63) and (65) below. Therefore, the modular conjugation can be considered a CPT operator. These results also hold in theories with braid group statistics. In the *absence* of braid group statistics, the CPT theorem can be made much more explicit [29,34], namely on the level of the field algebra. In fact, in this case the modular conjugation associated with $\mathcal{F}(W_1)$, multiplied with the so-called twist operator, is a CPT operator in the sense of Eq.s (4), (5) and (6).

In extending the Bisognano-Wichmann and CPT theorems to the case of braid group statistics, one encounters several difficulties. Since in this case there are no Wightman fields³, the original proofs of the CPT and Bisognano-Wichmann theorems do not work. Also the proof of modular covariance in [39] and the derivation of the explicit CPT theorem on the level of the field algebra from modular covariance [29,34] do not go through, on two accounts. Firstly, the “fractional spin” representations of the universal covering group of the Poincaré group do not share certain analyticity properties of the (half-) integer spin representations which have been used in [39]. This problem has been settled in the article [40], whose results have been used to prove the spin-statistics theorem for Plektons [42]. Secondly, the derivations of modular covariance in [39] and of the CPT theorem in [29,34] rely on the existence of an algebra \mathcal{F} of charge carrying field operators containing the observables \mathcal{A} as the sub-algebra of invariants under a (global) gauge symmetry and such that the vacuum is cyclic and separating for the local field algebras. Such a frame, which we shall call the Wick-Wightman-Wigner (WWW) scenario, always exists in the case of permutation group statistics [17], but does not exist in the case of non-Abelian braid group statistics. Then it is not even clear what the proper candidate for the Tomita operator should be. We use here a “pseudo”-Tomita operator which has already been proposed by Fredenhagen, Rehren and Schroer [23]. A major problem then is that one cannot use the algebraic relations of the modular objects among themselves and with respect to the field algebra, and with the representers of the translations. These relations are asserted by Tomita-Takesaki’s and Borchers’ [4] theorems, respectively, and enter crucially into the derivation of modular covariance and the CPT theorem in [29,34,39]. This problem has been partially settled in [41], where the algebraic relations of our pseudo-modular objects among themselves and with the translations have been analyzed.

In the present paper, we prove *pseudo-modular covariance*, namely that Eq. (3) holds with Δ^{it} standing for the pseudo-modular unitary group. We also show that the pseudo-modular conjugation already is a CPT operator in the sense of Eq.s (4), (5) and (6). Our line of argument parallels widely that of [39]. In the special case of Anyons, there does exist a WWW scenario and we show modular covariance in the usual sense.

The article is organized as follows. In Section 1 we specify our framework and assumptions in some detail. As our field algebra we shall use the reduced field bundle [23]. This is a C^* -algebra \mathcal{F} acting on a Hilbert space which contains, apart from the vacuum Hilbert space, subspaces corresponding to all charged sectors under consideration. It contains the observable algebra as the sub-algebra which leaves the vacuum Hilbert space invariant. However, in contrast to the field algebra in the permutation group statistics case it does not fulfil the WWW scenario. In particular the vacuum is not separating for the local algebras, and worse: *Every* field operator F which carries non-trivial charge satisfies $F^* \Omega = 0$. Correspondingly, there are no Tomita operators in the

³However there might be, in models, string-localized Wightman type fields in the sense of [43,55].

literal sense. This may be circumvented as proposed in [23]: There is a (non-involutive) pseudo-adjoint $F \mapsto F^\dagger$ on \mathcal{F} which coincides with the operator adjoint only for observables. The point here is that $F\Omega \mapsto F^\dagger\Omega$, $F \in \mathcal{F}(W_1)$, is a well-defined closable anti-linear operator.⁴ We define now $S \equiv J\Delta^{1/2}$ as in Eq. (2), with F^* replaced by F^\dagger , and call S , J and Δ^{it} the pseudo-Tomita operator, pseudo-modular conjugation and pseudo-modular unitary group, respectively, associated with $\mathcal{F}(W_1)$ — We add the word “pseudo” because S is not a Tomita operator in the strict sense (it is not even an involution). In Section 2, we express the pseudo-Tomita operator S in terms of a family of relative Tomita operators [56] associated with the observable algebra $\mathcal{A}(W_1)$ and certain suitably chosen pairs of states, and recall some algebraic properties of these objects established in [41]. Using these properties, we show that the pseudo-modular group associated with $\mathcal{F}(W_1)$ leaves this algebra invariant (Proposition 2.1), just as in the case of a genuine modular group. In Section 3 we derive single particle versions of pseudo-modular covariance and the CPT theorem (Corollary 3.3) from our assumption that the theory be purely massive (A1). This was already partially implicit in [42]. We then prove, in Section 4, that this property passes over from the single particle states to scattering states. Under the assumption of asymptotic completeness (A2), this amounts to pseudo-modular covariance of the field algebra. This is our main result, stated in Theorem 4.2. Since the \dagger -adjoint coincides with the operator adjoint on the observables, the restriction of S to the vacuum Hilbert space coincides with the (genuine) Tomita operator of the observables. We therefore have then modular covariance, in the usual sense, of the observables. As explained above, this also implies the CPT theorem on the level of observables. In Section 5 we make the CPT theorem explicit and show that the pseudo-modular conjugation of the field algebra $\mathcal{F}(W_1)$ is a CPT operator in the sense of Eq.s (4), (5)⁵ and (6) (Theorem 5.6). To this end, we use the mentioned CPT theorem on the observable level [29], as well as the algebraic properties of the “pseudo”-modular objects established in [41]. (The argument used in Section 5 of [39] via scattering theory cannot be used since it relies on the fact that the modular conjugation maps the algebra onto its commutant, which does not hold in the present case.⁶) In Section 6 we finally treat the case of Anyons, where there is known to be a field algebra \mathcal{F}_a in the WWW sense [38, 48]. In particular, the vacuum is cyclic and separating for $\mathcal{F}_a(W_1)$, allowing for the definition of a (genuine) Tomita operator associated with the wedge. Considering the genuine modular objects, we prove modular covariance (Theorem 6.2). We finally show that the modular conjugation, multiplied with an appropriate twist operator, is a CPT operator (Theorem 6.4). This extends the mentioned derivation of the CPT theorem in [29, 34] from Bosons and Fermions to Anyons. To achieve these results, we exhibit the anyonic field algebra \mathcal{F}_a as a sub-algebra of the reduced field bundle \mathcal{F} , and show that the corresponding Tomita operator of $\mathcal{F}_a(W_1)$ coincides with the pseudo-Tomita operator of $\mathcal{F}(W_1)$ (Lemma 6.3).

⁴To be precise, the “local” field algebras depend not only on spacetime regions such as W_1 , but also on certain paths in a sense to be specified in Section 1. In the definition of the pseudo-Tomita operator, W_1 must therefore be replaced by a path \tilde{W}_1 , see Eq. (22).

⁵In Eq. (5), C is now understood to be a path of space-like cones as explained in Section 1.

⁶On this occasion, I would like to rectify a minor error in the argument of [39, Section 5]. Namely, in Lemma 8 the modular conjugation, J_{W_1} , must be replaced by $Z^*J_{W_1}$, where Z is the twist operator, and twisted Haag duality for wedges [39, Eq. (1.4)] must be assumed. This does not influence the validity of its consequences, in particular of the CPT theorem (Proposition 9).

1 General Setting and Assumptions

Since we are aiming at model-independent results, we shall use the general framework of algebraic quantum field theory [1, 31], where only the physical principles of locality, covariance and stability are required, and formulated in mathematical terms in a quantum theoretical setting. We now specify this setting and make our assumptions precise.

Observable Algebra. The observables measurable in any given bounded spacetime region \mathcal{O} are modelled as (the self-adjoint part of) a von Neumann algebra $\mathcal{A}_0(\mathcal{O})$ of operators, such that observables localized in causally separated regions commute. These operators act in a Hilbert space \mathcal{H}_0 which carries a continuous unitary representation U_0 of the (proper orthochronous) Poincaré group P_+^\uparrow which acts geometrically correctly:

$$\text{Ad}U_0(g) : \mathcal{A}_0(\mathcal{O}) \rightarrow \mathcal{A}_0(g\mathcal{O}), \quad g \in P_+^\uparrow. \quad (7)$$

(By Ad we denote the adjoint action of unitaries.) To comply with the principle of stability or positivity of the energy, the energy-momentum spectrum of U_0 , namely the joint spectrum of the generators P_μ of the spacetime translations, is assumed to be contained in the forward light cone. The vacuum state corresponds to a unique (up to a factor) Poincaré invariant vector $\Omega_0 \in \mathcal{H}_0$. It has the Reeh-Schlieder property, namely it is cyclic and separating for every $\mathcal{A}_0(\mathcal{O})$. Since the vacuum state should be pure, the net of observables is assumed irreducible, $\cap_{\mathcal{O}} \mathcal{A}_0(\mathcal{O})' = \mathbb{C}\mathbf{1}$. For technical reasons, we also require that the observable algebra satisfy Haag duality for space-like cones and wedges.⁷ Namely, denoting by \mathcal{K} the class of space-like cones, their causal complements, and wedges, we require

$$\mathcal{A}_0(I') = \mathcal{A}_0(I)', \quad I \in \mathcal{K}. \quad (8)$$

($\mathcal{A}_0(I)$ is defined as the von Neumann algebra generated by all $\mathcal{A}_0(\mathcal{O})$, $\mathcal{O} \subset I$. The prime denotes the causal complement of a region on the left hand side, and the commutant of an algebra on the right hand side.) For the following discussion of charged sectors, it is convenient to enlarge the algebra of observables to the so-called universal algebra \mathcal{A} generated by isomorphic images $\mathcal{A}(I)$ of the $\mathcal{A}_0(I)$, $I \in \mathcal{K}$, see [20, 23, 28]. The family of isomorphisms $\mathcal{A}(I) \cong \mathcal{A}_0(I)$ extends to a representation π_0 of \mathcal{A} , the vacuum representation. We then have

$$\mathcal{A}_0(I) = \pi_0 \mathcal{A}(I),$$

and the vacuum representation is faithful and normal on the local⁸ algebras $\mathcal{A}(I)$.⁹ The adjoint action (7) of the Poincaré group on the local algebras lifts to a representation by automorphisms α_g of \mathcal{A} , $g \in P_+^\uparrow$, which acts geometrically correctly:

$$\begin{aligned} \text{Ad}U_0(g) \circ \pi_0 &= \pi_0 \circ \alpha_g, \\ \alpha_g : \mathcal{A}(I) &\rightarrow \mathcal{A}(gI). \end{aligned}$$

⁷A space-like cone with apex a is a region in Minkowski space of the form $C = a + \mathbb{R}^+ \mathcal{O}$, where \mathcal{O} is a double cone whose closure does not contain the origin. A wedge is a region which arises by a Poincaré transformation from W_1 , see Eq. (1).

⁸We call the algebras $\mathcal{A}(I)$ “local” although the regions I extend to infinity in some direction, just in distinction from the “global” algebra \mathcal{A} .

⁹However, π_0 is in general not faithful on the global algebra \mathcal{A} due to the existence of global intertwiners [23], see Footnote 10.

Charged Sectors. A superselection sector is an equivalence class of irreducible representations π of the algebra \mathcal{A}_0 of quasi-local observables, namely the C^* -algebra generated by all local observable algebras $\mathcal{A}_0(\mathcal{O})$. As a consequence of our Assumption A1, we shall deal only with representations which are localizable in space-like cones [12], i.e., equivalent to the vacuum representation when restricted to the causal complement of any space-like cone. Such representation uniquely lifts to a representation of the universal algebra \mathcal{A} . If Haag duality (8) holds, it is equivalent [14, 20] to a representation of the form $\pi_0 \circ \rho$ acting in \mathcal{H}_0 , where ρ is an endomorphism of \mathcal{A} localized in some specific region $C_0 \in \mathcal{K}$ in the sense that

$$\rho(A) = A \quad \text{if } A \in \mathcal{A}(C'_0). \quad (9)$$

The endomorphism ρ is further transportable to other space-like cones, which means that for every space-like cone C_1 and $I \in \mathcal{K}$ containing both C_0 and C_1 , there is a unitary $U \in \mathcal{A}(I)$ such that $\text{Ad}U \circ \rho$ is localized in C_1 .¹⁰ We shall call localized and transportable endomorphisms simply *localized morphisms*. We further assume the representation $\pi \cong \pi_0 \circ \rho$ to be covariant with positive energy. That means that there is a unitary representation U_ρ of the universal covering group \tilde{P}_+^\uparrow of the Poincaré group with spectrum contained in the forward light cone such that

$$\text{Ad}U_\rho(\tilde{g}) \circ \pi_0 \rho = \pi_0 \rho \circ \alpha_g, \quad g \in P_+^\uparrow, \quad (10)$$

where \tilde{g} is any element of \tilde{P}_+^\uparrow mapped onto g by the covering projection. Superselection sectors, namely equivalence classes of localizable representations of \mathcal{A}_0 , are in one-to-one correspondence with inner equivalence classes of localized morphisms of \mathcal{A} . They are the objects of a category whose three crucial structural elements are products, conjugation and sub-representations. More specifically, *products* $\rho_1 \rho_2 := \rho_1 \circ \rho_2$ of localized morphisms are again localized morphisms, leading to a composition of the corresponding sectors. A morphisms ρ localized in C is said to contain another such morphism τ as a *sub-representation* if there is a non-zero observable $T \in \mathcal{A}$, such that

$$\rho(A)T = T\tau(A) \quad \text{for all } A \in \mathcal{A}.$$

(If both ρ and τ are localized in a space-like cone C , then $T \in \mathcal{A}(C)$ by Haag duality.) An observable T satisfying this relation is called an intertwiner from τ to ρ . The set of all such intertwiners is denoted as $\text{Int}(\rho|\tau)$. They are the arrows between the objects ρ and τ . Arrows can be composed if they fit together and have adjoints. Namely: If $T \in \text{Int}(\rho|\tau)$ and $S \in \text{Int}(\tau|\sigma)$ then $T \circ S \in \text{Int}(\rho|\sigma)$; if $T \in \text{Int}(\rho|\tau)$ then $T^* \in \text{Int}(\tau|\rho)$. It follows that if τ is irreducible (i.e., the representation $\pi_0 \circ \tau$ of \mathcal{A} is irreducible), then $\text{Int}(\rho|\tau)$ is a Hilbert space with scalar product $\langle T, S \rangle$ being fixed by

$$\langle T, S \rangle \mathbf{1} := \pi_0(T^*S), \quad T, S \in \text{Int}(\rho|\tau).$$

There is also a product on the arrows, namely: if $T \in \text{Int}(\rho|\hat{\rho})$ and $S \in \text{Int}(\sigma|\hat{\sigma})$ then

$$T \times S := T \hat{\rho}(S) \equiv \rho(S)T \in \text{Int}(\rho\sigma|\hat{\rho}\hat{\sigma}). \quad (11)$$

¹⁰In $2 + 1$ dimensions, for every pair of causally separated space-like cones C_0, C_1 there are two topologically distinct ways to choose $I \supset C_0 \cup C_1$: Either one has to go clockwise from C_0 to C_1 within I , or anti-clockwise. This is the reason for the existence of global self-intertwiners in \mathcal{A} which are in the kernel of the vacuum representation, and makes the enlargement from \mathcal{A}_0 to \mathcal{A} necessary. It is also the reason for the occurrence of braid group statistics in $2 + 1$ dimensions.

As a consequence of our Assumption A1, all morphisms considered here have finite statistics [18], i.e. the so-called statistics parameter λ_ρ [14] is non-zero. This implies [15] the existence of a *conjugate* morphism $\bar{\rho}$ characterized, up to equivalence, by the fact that the composite sector $\pi_0 \bar{\rho} \rho$ contains the vacuum representation π_0 precisely once. Thus there is a unique, up to a factor, intertwiner $R_\rho \in \mathcal{A}(C_0)$ satisfying $\bar{\rho} \rho(A) R_\rho = R_\rho A$ for all $A \in \mathcal{A}$. The conjugate $\bar{\rho}$ shares with ρ the properties of covariance (10), finite statistics, and localization (9) in some space-like cone which we choose to be C_0 . Using the normalization convention of [15, Eq. (3.14)], namely $R_\rho^* R_\rho = |\lambda_\rho|^{-1} \mathbb{1}$, the positive linear endomorphism ϕ_ρ of \mathcal{A} defined as

$$\phi_\rho(A) = |\lambda_\rho| R_\rho^* \bar{\rho}(A) R_\rho \quad (12)$$

is the unique left inverse [12,15] of ρ . In the present situation of three-dimensional space-time, the statistics parameter λ_ρ may be a complex non-real number, corresponding to braid group statistics. We admit the case when its modulus is different from one (namely when ρ is not surjective), corresponding to non-Abelian braid group statistics.

Field Algebra. From the observable algebra and a set of relevant sectors a field algebra can be constructed in various ways, see for example [23, 25–27, 37, 49]. Some of these constructions have a quantum group or a more general structure playing the role of a global gauge group, however none of them fulfils the WWW scenario. Unfortunately, most constructions work only for models with a finite set of charges, whereas we wish to consider here an arbitrary (though countable) number of charges. We choose as field algebra the reduced field bundle proposed in [23], which in turn has been based on the field bundle of [15]. We start with a countable collection Σ of pairwise inequivalent localized, covariant, irreducible morphisms with finite statistics, one from each relevant sector, which is stable under conjugations and composition with subsequent reduction, and contains the identity morphism ι . We take all morphisms to be localized in the same space-like cone C_0 , which we choose to be contained in W_1 . The space of all relevant states is described by the Hilbert space

$$\mathcal{H} := \bigoplus_{\rho \in \Sigma} \mathcal{H}_\rho, \quad \mathcal{H}_\rho = \mathcal{H}_0. \quad (13)$$

We shall denote elements of \mathcal{H}_ρ as (ρ, ψ) . For each $\rho \in \Sigma$ there is a unitary representation U_ρ of \tilde{P}_+^\dagger acting in \mathcal{H}_0 . (For $\rho = \iota$, we take $U_\iota \equiv U_0$.) This gives rise to the direct sum representation U on \mathcal{H}

$$U(\tilde{g})(\rho, \psi) := (\rho, U_\rho(\tilde{g})\psi). \quad (14)$$

The vacuum vector

$$\Omega := (\iota, \Omega_0) \in \mathcal{H}_\iota$$

is invariant under this representation. The observables act in \mathcal{H} via the direct sum of all relevant representations $\pi_0 \circ \rho =: \pi_0 \rho$,

$$\pi(A)(\rho, \psi) := (\rho, \pi_0 \rho(A)\psi).$$

The idea of a charge carrying field is that it should add a certain charge ρ_c to a given state $\psi \in \mathcal{H}_{\rho_s}$. But since the product morphism $\rho_s \rho_c$ is, in general, not contained in the chosen set of irreducible morphisms, the new state must be projected onto an irreducible

sub-representation $\rho_r \in \Sigma$ of $\rho_s \rho_c$. (The subscripts s, c, r stand for “source”, “charge” and “range”, respectively.) This idea is realized as follows [22, 23]. Given any three $\rho_s, \rho_c, \rho_r \in \Sigma$ such that $\rho_s \rho_c$ contains ρ_r as a sub-representation, the corresponding intertwiner space $\text{Int}(\rho_s \rho_c | \rho_r)$ has a certain finite [23] dimension N . We choose an orthonormal basis, i.e., a collection $T_i \in \mathcal{A}(C_0)$, $i = 1, \dots, N$, satisfying

$$T_i^* T_j = \delta_{ij} \mathbf{1}, \quad \sum_{i=1}^N T_i T_i^* = \mathbf{1}_{\rho_s \rho_c},$$

where $\mathbf{1}_{\rho_s \rho_c}$ is the unit in the algebra $\text{Int}(\rho_s \rho_c | \rho_s \rho_c)$. Following [46], we shall call the multi-index

$$e := (\rho_s, \rho_c, \rho_r, i)$$

a “superselection channel” of type (ρ_s, ρ_c, ρ_r) , and denote any one of the T_i from above generically as

$$T_e \in \text{Int}(\rho_s \rho_c | \rho_r).$$

We shall also call $s(e) := \rho_s$, $c(e) := \rho_c$ and $r(e) := \rho_r$ the source, charge and range of e , respectively. If $s(e)$ or $c(e) = \iota$, we choose $T_e = \mathbf{1}$. The charge carrying fields are now defined as follows. Given e of type (ρ_s, ρ_c, ρ_r) and $A \in \mathcal{A}$, $F(e, A)$ is the operator in \mathcal{H} defined by

$$F(e, A)(\rho, \psi) := \delta_{\rho_s, \rho} (\rho_r, \pi_0(T_e^* \rho(A)) \psi).$$

Heuristically, this describes the action of A in the background charge ρ , addition of the charge ρ_c and subsequent projection onto \mathcal{H}_{ρ_r} via the intertwiner T_e^* . The norm-closed linear span of all these operators,

$$\mathcal{F} := \left(\bigoplus_e \{F(e, A), A \in \mathcal{A}\} \right)^-,$$

where the sum goes over all superselection channels e , is closed under multiplication and will be called the field algebra. It is in fact a C^* sub-algebra of $\mathcal{B}(\mathcal{H})$ [22]. It contains the (representation π of the) observable algebra, namely

$$\pi(A) = \sum_{e: c(e) = \iota} F(e, A).$$

Localization. Fields are localizable to the same extent to which the charges are localizable which they carry, namely in unbounded regions in the class \mathcal{K} . In order to have any definite space-like commutation relations, the fields need to carry some supplementary information in addition to the localization region, due to the existence of global intertwiners (see Footnote 10). The possibility we choose is to consider paths in \mathcal{K} starting from our fixed reference cone C_0 .¹¹ By a *path* in \mathcal{K} from C_0 to a region $I \in \mathcal{K}$ (or “ending at” I) we mean a finite sequence (I_0, \dots, I_n) of regions in \mathcal{K} with $I_0 = C_0$, $I_n = I$, such that either $I_{k-1} \subset I_k$ or $I_{k-1} \supset I_k$ for $k = 1, \dots, n$. Given a path $(C_0 = I_0, I_1, \dots, I_n = I)$ and a morphism $\rho \in \Sigma$ there are unitaries $U_k \in \mathcal{A}(I_{k-1} \cup I_k)$ such that $\rho_k := \text{Ad}(U_k \cdots U_1) \circ \rho$ is localized in I_k . We shall call $U := U_n \cdots U_1$ a

¹¹Two other possibilities are: To introduce a reference space-like cone from which all allowed localization cones have to keep space-like separated (this cone playing the role of a “cut” in the context of multi-valued functions) [12]; or a cohomology theory of nets of operator algebras as introduced by Roberts [50–52].

charge transporter for ρ along the path (I_0, \dots, I_n) . Now a field operator $F(e, A)$ with $c(e) = \rho$, is said to be *localized* along a path in \mathcal{K} ending at I if there is a charge transporter U for ρ along the path such that

$$UA \in \mathcal{A}(I).$$

This localization concept clearly depends only on the homotopy classes (in an obvious sense [23]) of paths. We shall denote the homotopy class of a path ending at I by \tilde{I} , and the set of all such (classes of) paths by $\tilde{\mathcal{K}}$. The field operators localized in a given path \tilde{I} generate a sub-algebra of \mathcal{F} which we denote by $\mathcal{F}(\tilde{I})$. The vacuum Ω is cyclic for the local fields, i.e. for any path \tilde{I} there holds

$$(\mathcal{F}(\tilde{I}) \Omega)^- = \mathcal{H}. \quad (15)$$

Note, however, that Ω is not separating for the local¹² field algebras $\mathcal{F}(\tilde{I})$, since every field with non-trivial source annihilates the vacuum.

We now give an alternative description of $\tilde{\mathcal{K}}$ in the spirit of the “string-localized” quantum fields proposed in [43], which will be useful in the sequel. It is based on the observation that a space-like cone C is characterized by its apex $a \in \mathbb{R}^3$ and the space-like directions contained in C . Namely, let H be the manifold of space-like directions,

$$H := \{r \in \mathbb{R}^3, r \cdot r = -1\}.$$

The set of space-like directions contained in C is $C^H := (C - a) \cap H$, and there holds $C = a + \mathbb{R}^+ C^H$. C^H is in fact a double cone in H .¹³ A similar consideration holds for causal complements of space-like cones and wedge regions (except that the apex of a wedge is fixed only modulo translations along its edge). We can therefore identify regions in \mathcal{K} with regions of the form

$$\{a\} \times I^H \subset \mathbb{R}^3 \times H, \quad (16)$$

where I^H is a double cone, a causal complement thereof, or a wedge, in H . Let us denote the class of such regions by \mathcal{K}^H . Regions in \mathcal{K}^H are simply connected, whereas H itself has fundamental group \mathbb{Z} . Thus the portion of the universal covering space of H over a region I^H in \mathcal{K}^H consists of a countable infinity of copies (“sheets”) of I^H . We shall generically denote such a sheet over I^H by \tilde{I}^H , and we denote by $\tilde{\mathcal{K}}^H$ the class of such sheets. We identify the universal covering space \tilde{H} of H with homotopy classes of paths in H starting at some fixed reference direction r_0 , which we assume to be contained in the reference cone C_0 . (A sheet \tilde{I}^H is canonically homeomorphic to I^H , but contains in addition the information of a winding number distinguishing it from the other sheets over I^H , see Figure 1.) We now identify paths $\tilde{I} \in \tilde{\mathcal{K}}$ with regions of the form

$$\{a\} \times \tilde{I}^H \subset \mathbb{R}^3 \times \tilde{H}, \quad (17)$$

as follows. Given a path $(I_0 = C_0, I_1, \dots, I_n = I)$ in \mathcal{K} , pick a path $\gamma = \gamma_n * \dots * \gamma_0$ in H from r_0 to some r contained in I , and points a_0, \dots, a_n in \mathbb{R}^3 with $a_0 = 0$ and $a_n = a =$ apex of I , such that $\gamma_k(t) \in I_k - a_k$ for $t \in [0, 1]$, $k = 0, \dots, n$. Then we associate

¹²Again, we call the algebras $\mathcal{F}(\tilde{I})$ “local” just in distinction to the “global” algebra \mathcal{F} .

¹³This is so because the boundary of $C - a$ consists of 4 (pieces of) light-like planes through the origin. The intersection of such a plane with H is a light-like geodesic in H [44, proof of Prop. 28]. Thus, C^H is bounded by 4 light-like geodesics emanating from two time-like separated points, and therefore is a double cone in the two-dimensional spacetime H .

with (I_0, \dots, I_n) the region (17), where \tilde{I}^H is the unique sheet over $I^H = (I - a) \cap H$ which contains the homotopy class of γ . Different paths γ lead to the same sheet, and the sheet depends only on the “homotopy class” (in the sense of [23]) of (I_0, \dots, I_n) . Therefore the above prescription defines a one-to-one correspondence between $\tilde{\mathcal{K}}$ and $\mathbb{R}^3 \times \tilde{\mathcal{K}}^H$, which shall be used to identify them. In this identification, the covering space aspect of $\tilde{\mathcal{K}}$ shows up in “accumulated angles”, endowing $\tilde{\mathcal{K}}$ with a partial order relation. Namely, given $\tilde{I}_i = \{a_i\} \times \tilde{I}_i^H$ with I_1 and I_2 space-like separated, we shall write

$$\tilde{I}_1 < \tilde{I}_2$$

if for any $[\gamma_i] \in \tilde{I}_i^H$ there holds $\int_{\gamma_1} d\theta < \int_{\gamma_2} d\theta$, where $d\theta$ denotes the angle one-form in a fixed Lorentz frame. (This is well-defined since the last relation is independent of the representants of $[\gamma_i]$ and of the Lorentz frame.)

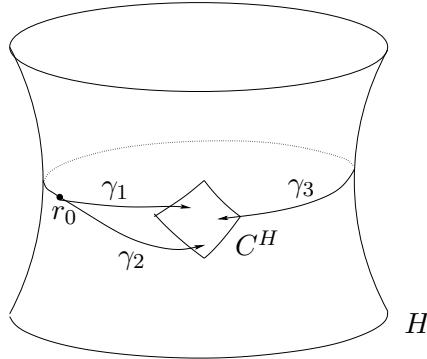


Figure 1: The (classes of the) paths γ_1 and γ_2 lie in the same sheet, say \tilde{C}^H , over C^H . The path γ_3 lies in a different sheet, which is mapped by a 2π rotation onto \tilde{C}^H .

Covariance. The adjoint action of the representation U of \tilde{P}_+^\uparrow leaves the field algebra invariant, more specifically [15]:

$$U(\tilde{g}) F(e, A) U(\tilde{g})^* = (e, Y_\rho(\tilde{g}) \alpha_g(A)), \quad (18)$$

where $\rho = s(e)$ is the charge of e , and the so-called cocycle $Y_\rho(\tilde{g}) \in \mathcal{A}$ is characterized by

$$\pi_0(Y_\rho(\tilde{g})) = U_\rho(\tilde{g}) U_0(g)^*. \quad (19)$$

The adjoint action on the fields is geometrically correct, i.e.,

$$\text{Ad}U(\tilde{g}) : \mathcal{F}(\tilde{I}) \rightarrow \mathcal{F}(\tilde{g} \cdot \tilde{I}).$$

Here, $\tilde{g} \cdot \tilde{I}$ denotes the natural action of the universal covering of the Poincaré group on $\tilde{\mathcal{K}}$, defined as follows. Let $\tilde{g} = (x, \tilde{\lambda})$, where x is a spacetime translation and $\tilde{\lambda}$ is an element of the universal covering group \tilde{L}_+^\uparrow of the Lorentz group, projecting onto $\lambda \in L_+^\uparrow$. Then

$$(x, \tilde{\lambda}) \cdot (\{a\} \times \tilde{I}^H) := \{x + \lambda a\} \times \tilde{\lambda} \cdot \tilde{I}^H, \quad (20)$$

where $\tilde{\lambda} \cdot \tilde{I}^H$ denotes the lift of the action of the Lorentz group on H to the respective universal covering spaces. The rotations about integer multiples of 2π do not act trivially, but rather coincide with the action of the fundamental group, \mathbb{Z} , on the

universal covering space of H . Namely, they change winding numbers, see Figure 1. Related to this, we define the *relative winding number* $N(\tilde{I}_2, \tilde{I}_1)$ of \tilde{I}_2 w.r.t. \tilde{I}_1 to be the unique integer n such that

$$\tilde{r}(2\pi n) \cdot \tilde{I}_1 < \tilde{I}_2 < \tilde{r}(2\pi(n+1)) \cdot \tilde{I}_1,$$

where $\tilde{r}(\cdot)$ denotes the rotation subgroup in \tilde{L}_+^\uparrow . See Fig. 2 for an example. (Note that this number is independent of the choice of reference direction r_0 .)

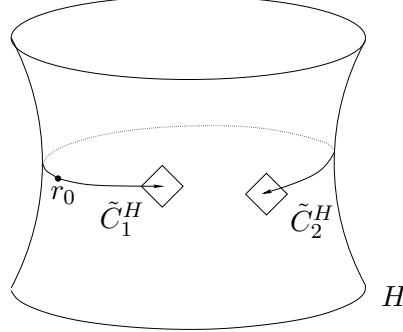


Figure 2: \tilde{C}_1 and \tilde{C}_2 have relative winding number $N(\tilde{C}_2, \tilde{C}_1) = -1$.

Pseudo-Adjoint. Let \mathcal{F}_ι be the Banach space generated by field operators $F(e, A) \in \mathcal{F}$ which have trivial source, $s(e) = \iota$, i.e., which have e of the form (ι, ρ, ρ) . For such e , we define an adjoint channel $\bar{e} := (\iota, \bar{\rho}, \bar{\rho})$. Following [23], we define a pseudo-adjoint $F \mapsto F^\dagger$ on the space \mathcal{F}_ι by

$$F(e, A)^\dagger := F(\bar{e}, \bar{\rho}(A^*)R_\rho), \quad e = (\iota, \rho, \rho).$$

This pseudo-adjoint does *not* coincide with the operator adjoint (with respect to which \mathcal{F} is a C^* -algebra). In fact, it is not an involution, but rather satisfies

$$(F^\dagger)^\dagger = \chi_\rho F \tag{21}$$

if F has charge ρ . The number χ_ρ is a root of unity in a self-conjugate sector ($\bar{\rho} \simeq \rho$), intrinsic to the sector, while in all other sectors R_ρ and $R_{\bar{\rho}}$ may be chosen so that $\chi_\rho = 1$ [23, Eq. (3.2)]. (Those self-conjugate sectors with $\chi_\rho = -1$ are called pseudo-real sectors.) The adjoint preserves localization, i.e. leaves invariant the spaces of local fields with trivial source, $\mathcal{F}_\iota(\tilde{I}) := \mathcal{F}_\iota \cap \mathcal{F}(\tilde{I})$:

$$(\mathcal{F}_\iota(\tilde{I}))^\dagger = \mathcal{F}_\iota(\tilde{I}).$$

Finally, the adjoint is preserved by Poincaré transformations:

$$(U(\tilde{g})FU(\tilde{g})^*)^\dagger = U(\tilde{g})F^\dagger U(\tilde{g})^*$$

for all $\tilde{g} \in \tilde{P}_+^\uparrow$ and $F \in \mathcal{F}_\iota$. Due to the faithfulness of $\pi_0\rho$ on the local algebras $\mathcal{A}(I)$, $F^\dagger\Omega = 0$ implies $F = 0$ for $F \in \mathcal{F}(\tilde{I})$. This allows for the definition of our pseudo-Tomita operator [23]

$$S : F\Omega \mapsto F^\dagger\Omega, \quad F \in \mathcal{F}_\iota(\tilde{W}_1). \tag{22}$$

Here, \tilde{W}_1 is a path ending at W_1 which will be specified in Eq. (28) below.

Braid Group Statistics. For every pair of localized morphisms ρ, σ in Σ there is a local unitary intertwiner $\varepsilon(\rho, \sigma) \in \text{Int}(\sigma\rho|\rho\sigma)$, the so-called statistics operator. The family of statistics operators satisfies the braid relations [23, Eq. (2.6)] and determines the statistics of fields, as follows. Let C_1 and C_2 be causally separated, and let \tilde{C}_i be paths ending at C_i with relative winding number $N(\tilde{C}_2, \tilde{C}_1) = n$, and let $F(e_1, A_1) \in \mathcal{F}(\tilde{C}_1)$ and $F(e_2, A_2) \in \mathcal{F}(\tilde{C}_2)$ be two fields with superselection channels e_1 of type (α, ρ_1, β) and e_2 of type (β, ρ_2, γ) , where $\alpha, \beta, \gamma, \rho_i \in \Sigma$. Then there holds the commutation relation [23, Prop. 5.9]

$$F(e_2, A_2) F(e_1, A_1) = \sum_{\delta, i_1, i_2} R(\delta, e_1, e_2, n) F(\hat{e}_1, A_1) F(\hat{e}_2, A_2). \quad (23)$$

Here $\hat{e}_1 = (\delta, \rho_1, \gamma, i_1)$ and $\hat{e}_2 = (\alpha, \rho_2, \delta, i_2)$, and the sum goes over all morphisms δ which are contained in the product representation $\alpha \circ \rho_2$. The numbers $R(\cdot)$ are given by

$$R(\delta, e_1, e_2, n) = \left(\frac{\omega_\alpha \omega_\gamma}{\omega_\beta \omega_\delta} \right)^n \pi_0(T_{e_2}^* T_{e_1}^* \alpha(\varepsilon(\rho_2, \rho_1)) T_{\hat{e}_2} T_{\hat{e}_1}). \quad (24)$$

The vacuum expectation values (26) of these commutation relations are already determined by the statistics phases. The statistics parameter λ_ρ and statistics phase ω_ρ of a sector $[\rho]$ are defined by the relations

$$\phi_\rho(\varepsilon(\rho, \rho)) = \lambda_\rho \mathbf{1}, \quad \omega_\rho = \frac{\lambda_\rho}{|\lambda_\rho|}, \quad (25)$$

respectively. (They depend only on the equivalence class of ρ .) Suppose now that C_1 and C_2 are causally separated, and \tilde{C}_1 and \tilde{C}_2 have relative winding number $N(\tilde{C}_2, \tilde{C}_1) = -1$, see Figure 2 for an example. Then for $F_1 = F(e, A_1)$ and $F_2 = F(e, A_2) \in \mathcal{F}_\ell(\tilde{C}_i)$ with $e = (\iota, \rho, \rho)$ there holds

$$(F_2 \Omega, F_1 \Omega) = \omega_\rho (F_1^\dagger \Omega, F_2^\dagger \Omega), \quad (26)$$

see e.g. [15, Eq. (6.5)] and [42, Lemma A.1]. Of course $\omega_\rho = \pm 1$ corresponds to Bosons/Fermions, while the generic case corresponds to braid group statistics. Note that the hypothesis on the relative winding number under which Eq. (26) holds is not symmetric in \tilde{C}_1 and \tilde{C}_2 . Without this condition, Eq. (26) would imply $\omega_\rho \omega_{\bar{\rho}} = 1$. But ω_ρ and $\omega_{\bar{\rho}}$ are known to coincide [25], hence Eq. (26) would be self-consistent only for $\omega_\rho = \pm 1$, excluding braid group statistics.

Assumptions. We shall assume that the theory is purely massive, and that asymptotic completeness holds. By purely massive, we mean that the set Σ of relevant sectors is generated by a set of elementary charges, which correspond to massive particles.

A covariant representation is called a *massive single particle representation* if its mass spectrum¹⁴ contains a strictly positive eigenvalue (the mass of the corresponding particle type), isolated from the rest of the mass spectrum in its sector by a mass gap (implementing the idea that there are no massless particles in the model). We also assume that there are only finitely many particle types in a given sector with a given mass, and that these have the same spin. We thus make the

¹⁴By mass spectrum we mean the spectrum of the mass operator $P_\mu P^\mu$.

Assumption A1 (Massive particle spectrum.) *There is a finite subset*

$$\Sigma^{(1)} \subset \Sigma$$

of morphisms corresponding to massive single particle representations, which generates Σ . (In other words, Σ is exhausted by composition and subsequent reduction of morphisms in $\Sigma^{(1)}$.) For each $\rho \in \Sigma^{(1)}$, the restriction of the representation U_ρ to the eigenspace of the corresponding mass value¹⁵ m_ρ is a finite multiple of an irreducible representation.

(Note that Σ is countable but may be infinite.) It is gratifying that this assumption, together with Haag duality (8), implies that all relevant sectors really have as representatives localized morphisms [12] with finite statistics [18] as assumed in our framework. Our second assumption is that the theory can be completely interpreted in terms of multi-particle states:

Assumption A2 (Asymptotic Completeness.) *The scattering states span the entire Hilbert space \mathcal{H} .*

(We shall sketch in Section 4 the Haag-Ruelle construction of scattering states from single particle states in the setting of the reduced field bundle for Plektons.)

2 Algebraic Properties of the Pseudo-Modular Objects

As a first step, we discuss algebraic properties of the pseudo-modular objects which are independent of our special assumptions A1 and A2, and are analogous to properties of genuine modular objects. In particular, we show that the (adjoint action of the) pseudo-modular group leaves the field algebra of the wedge invariant, and point out that Borchers' commutation relations between the modular objects and the translations also hold in the present case. To this end, we exhibit the pseudo-Tomita operator S as a family of *relative* Tomita operators [56] S_ρ , $\rho \in \Sigma$, associated with the observable algebra and certain suitably chosen pairs of states. We then use results obtained in a recent article [41] by the author on the relative modular objects.

Recall that S maps each \mathcal{H}_ρ to $\mathcal{H}_{\bar{\rho}}$, and therefore corresponds to a family of operators S_ρ acting in \mathcal{H}_0 via

$$S(\rho, \psi) =: (\bar{\rho}, S_\rho \psi). \quad (27)$$

In order to calculate each operator S_ρ explicitly, we first have to specify the path \tilde{W}_1 from C_0 to W_1 which enters in its definition (22). For simplicity, we shall take the reference cone C_0 to be properly contained in W_1 (i.e., its closure is contained in W_1), and define \tilde{W}_1 to be the (class of the) path

$$\tilde{W}_1 := (C_0, W_1). \quad (28)$$

Then, $F(e, A)$ is in $\mathcal{F}_\ell(\tilde{W}_1)$ if and only if A is in $\mathcal{A}(W_1)$ and e is of the form $e = (0, \rho, \rho)$ for some $\rho \in \Sigma$, in which case $T_e = R_\rho$. Thus the definition (22) reads explicitly

$$S_\rho \pi_0(A) \Omega_0 = \pi_0[\bar{\rho}(A^*) R_\rho] \Omega_0, \quad A \in \mathcal{A}(W_1).$$

¹⁵For simplicity, we shall assume that there is only one mass eigenvalue in each sector, but our results still hold if no restriction is imposed on the number of (isolated) mass values in each sector.

In the special case $\rho = \iota$, this is just the Tomita operator of the observables, which we shall denote by $S_0 \equiv J_0 \Delta_0^{1/2}$. In the general case $\rho \neq \iota$, S_ρ is the relative Tomita operator associated with the algebra $\mathcal{A}(W_1)$ and the pair of states $\omega_0 := (\Omega_0, \pi_0(\cdot)\Omega_0)$ (the vacuum state) and the positive functional

$$\varphi_\rho := |\lambda_\rho|^{-1} \omega_0 \circ \phi_\rho = (R_\rho \Omega_0, \pi_0 \bar{\rho}(\cdot) R_\rho \Omega_0).$$

We denote again the polar decomposition by $S_\rho = J_\rho \Delta_\rho^{1/2}$ and call Δ_ρ^{it} and J_ρ the relative modular group and conjugation, respectively. It is known in relative Tomita-Takesaki theory [56] that the operator $\Delta_\rho^{it} \Delta_0^{-it}$ is in $\pi_0 \mathcal{A}(W_1)$ for $t \in \mathbb{R}$, giving rise to a family of unitaries

$$Z_\rho(t) := \pi_0^{-1}(\Delta_\rho^{it} \Delta_0^{-it}) \in \mathcal{A}(W_1) \quad (29)$$

known as the Connes cocycle $(D\varphi_\rho : D\omega_0)_t$ with respect to the pair of weights ω_0 and φ_ρ . Starting from the Connes cocycle, the algebraic properties of the relative modular objects have been analyzed in [41]. It has been shown there that the relative modular unitary group Δ_ρ^{it} and conjugation J_ρ satisfy the implementation properties

$$\begin{aligned} \text{Ad} \Delta_\rho^{it} \circ \pi_0 \rho &= \pi_0 \rho \circ \sigma_t, \\ \text{Ad} J_\rho \circ \pi_0 \rho &= \pi_0 \bar{\rho} \circ \alpha_j \end{aligned} \quad (30)$$

on $\mathcal{A}(W_1) \cup \mathcal{A}(W'_1)$. Here, σ_t is the modular group associated with $\mathcal{A}(W_1)$ and the vacuum state [8], and α_j is an anti-isomorphism from $\mathcal{A}(W_1)$ onto $\mathcal{A}(W'_1)$ and vice versa. These are characterized by the fact that there holds

$$\begin{aligned} \text{Ad} \Delta_0^{it} \circ \pi_0 &= \pi_0 \circ \sigma_t \\ \text{Ad} J_0 \circ \pi_0 &= \pi_0 \circ \alpha_j \end{aligned} \quad (31)$$

on $\mathcal{A}(W_1) \cup \mathcal{A}(W'_1)$. Furthermore, Borchers' commutation relations have been shown [41] to hold between the relative modular objects and the translations, namely for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^3$ there holds

$$\Delta_\rho^{it} U_\rho(x, 1) \Delta_\rho^{-it} = U_\rho(\lambda_1(-2\pi t)x, 1), \quad (32)$$

$$J_\rho U_\rho(x, 1) J_\rho^{-1} = U_{\bar{\rho}}(jx, 1). \quad (33)$$

Finally, the algebraic relations among the relative modular objects, also established in [41], are completely analogous to the case of the genuine ones:

$$J_\rho \Delta_\rho^{it} J_\rho^{-1} = \Delta_{\bar{\rho}}^{it}, \quad (34)$$

$$J_\rho J_{\bar{\rho}} = \chi_\rho \mathbf{1}, \quad (35)$$

where χ_ρ are the factors defined after Eq. (21). Note that these equations (or Eq. (21)) imply that the pseudo-Tomita operator S of the field algebra is not an involution, but rather satisfies

$$S^2 \subset \chi := \sum_{\rho \in \Sigma} \chi_\rho E_\rho. \quad (36)$$

Using relative Tomita-Takesaki theory and the fact that the pseudo-modular objects are related to the relative ones by

$$J(\rho, \psi) = (\bar{\rho}, J_\rho \psi), \quad \Delta^{it}(\rho, \psi) = (\rho, \Delta_\rho^{it} \psi) \quad (37)$$

due to uniqueness of the polar decomposition, we shall now calculate the adjoint action of the pseudo-modular group on the fields localized in \tilde{W}_1 . It turns out that this action leaves $\mathcal{F}(\tilde{W}_1)$ invariant — a non-trivial fact which enters crucially in the calculation of Δ^{it} on scattering states in Section 4. For completeness' sake we also show that the action of the pseudo-modular group commutes with the pseudo-adjoint \dagger .

Proposition 2.1 *The adjoint action of the pseudo-modular group leaves the field algebra $\mathcal{F}(\tilde{W}_1)$ associated to the wedge invariant, and commutes with the pseudo-adjoint \dagger on $\mathcal{F}_t(\tilde{W}_1)$:*

$$\Delta^{it} \mathcal{F}(\tilde{W}_1) \Delta^{-it} = \mathcal{F}(\tilde{W}_1), \quad (38)$$

$$(\Delta^{it} F \Delta^{-it})^\dagger = \Delta^{it} F^\dagger \Delta^{-it}, \quad (39)$$

$F \in \mathcal{F}_t(\tilde{W}_1)$. Specifically, if e is a superselection channel with $c(e) = \rho$, then

$$\Delta^{it} F(e, A) \Delta^{-it} = F(e, Z_\rho(t) \sigma_t^0(A)). \quad (40)$$

It is interesting to note the resemblance of Eq.s (29) and (40) with Eq.s (19) and (18), respectively.

Proof. We shall use two facts about the Connes cocycle (29) established by Longo in the present context. Namely, on $\mathcal{A}(W_1) \cup \mathcal{A}(W'_1)$ there holds [36, Prop. 1.1]

$$\text{Ad}Z_\rho(t) \circ \sigma_t \circ \rho = \rho \circ \sigma_t, \quad (41)$$

and for any intertwiner T^* from ρ_r to $\rho_s \rho_c$ there holds [36, Props. 1.3, 1.4]

$$T^* \rho_s(Z_{\rho_c}(t)) Z_{\rho_s}(t) = Z_{\rho_r}(t) \sigma_t^0(T^*). \quad (42)$$

For the proof of the proposition, let A be in $\mathcal{A}(W_1)$ and e be of type (ρ_s, ρ_c, ρ_r) . There holds

$$\begin{aligned} \Delta^{it} F(e, A) \Delta^{-it} (\rho_s, \psi) &= (\rho_r, \Delta_{\rho_r}^{it} \pi_0(T_e^* \rho_s(A)) \Delta_{\rho_s}^{-it} \psi) \\ &= (\rho_r, \pi_0(T_e^* \rho_s(Z_{\rho_c})) \Delta_{\rho_s}^{it} \pi_0 \rho_s(A) \Delta_{\rho_s}^{-it} \psi) \\ &= (\rho_r, \pi_0(T_e^* \rho_s(Z_{\rho_c})) \pi_0 \rho_s(\sigma_t^0(A)) \psi) \\ &= F(e, Z_{\rho_c}(t) \sigma_t^0(A)) (\rho_s, \psi). \end{aligned}$$

In the third equation we have used Eq. (30), and in the second one we have used that

$$\Delta_{\rho_r}^{it} \pi_0(T^*) = \pi_0(T^* \rho_s(Z_{\rho_c}(t)) \Delta_{\rho_s}^{it}),$$

which is a consequence of Eq. (42). This proves the explicit formula (40). Since $Z_{\rho_c}(t)$ and $\sigma_t(A)$ both are in $\mathcal{A}(W_1)$, this also shows invariance (38) of $\mathcal{F}(\tilde{W}_1)$. To prove Eq. (39), let $e = (\iota, \rho, \rho)$ and $\bar{e} := (\iota, \bar{\rho}, \bar{\rho})$. Then

$$\begin{aligned} (\Delta^{it} F(e, A) \Delta^{-it})^\dagger &= (\bar{e}, \bar{\rho}[\sigma_t^0(A^*) Z_\rho(t)^*] R_\rho), \\ \Delta^{it} F(e, A)^\dagger \Delta^{-it} &= (\bar{e}, Z_{\bar{\rho}}(t) \sigma_t^0[\bar{\rho}(A^*) R_\rho]). \end{aligned}$$

But Eq.s (41) and (42) imply that

$$\bar{\rho}[\sigma_t^0(A) Z_\rho(t)^*] R_\rho = Z_{\bar{\rho}}(t) \sigma_t^0 \bar{\rho}(A) Z_{\bar{\rho}}(t)^* \bar{\rho}(Z_\rho(t)^*) R_\rho = Z_{\bar{\rho}}(t) \sigma_t^0 [\bar{\rho}(A) R_\rho].$$

This shows Eq. (39) and completes the proof. \square

3 Modular Covariance and CPT operator on the Single Particle Space

As a first step, we prove single-particle versions of the Bisognano-Wichmann and the CPT theorems. For ρ in the set $\Sigma^{(1)}$ of single particle charges, let E_ρ be the projection from \mathcal{H} onto \mathcal{H}_ρ , and $E_\rho^{(1)}$ the projection from \mathcal{H} onto the eigenspaces of the mass operator in \mathcal{H}_ρ (corresponding to the isolated eigenvalues in the sector ρ). We denote by $E^{(1)}$ the sum of all $E_\rho^{(1)}$, where ρ runs through $\Sigma^{(1)}$, and call the range of $E^{(1)}$ the single particle space.

Borchers' commutation relations (32), (33) imply that the pseudo-modular unitary group and the pseudo-modular conjugation commute with the mass operator. Hence the pseudo-Tomita operator S commutes with $E^{(1)}$. Let us denote the corresponding restriction by

$$S^{(1)} := S E^{(1)}.$$

Similarly, the representation $U(\tilde{P}_+^\dagger)$ leaves $E^{(1)}\mathcal{H}$ invariant, giving rise to the sub-representation

$$U^{(1)}(g) := U(g) E^{(1)},$$

and one may ask if modular covariance holds on $E^{(1)}\mathcal{H}$. We show in this section that this is indeed the case, the line of argument being as follows. Let K denote the generator of the unitary group of 1-boosts, $U^{(1)}(\lambda_1(t)) = \exp(itK)$. We exhibit in Eq. (46) below an anti-unitary “CPT -operator” $U^{(1)}(j)$ representing the reflexion j on $E^{(1)}\mathcal{H}$, and show that $S^{(1)}$ coincides with the “geometric” involution

$$S_{\text{geo}}^{(1)} := U^{(1)}(j) e^{-\pi K} \quad (43)$$

up to a unitary operator which commutes with the representation $U^{(1)}$ of \tilde{P}_+^\dagger . By uniqueness of the polar decomposition, this will imply modular covariance on $E^{(1)}\mathcal{H}$, namely $U^{(1)}(\lambda_1(-2\pi t)) \equiv \exp(-2\pi itK) = \Delta^{it} E^{(1)}$.

We begin by exploiting our knowledge about $U^{(1)}(\tilde{P}_+^\dagger)$. By assumption, for each $\rho \in \Sigma^{(1)}$ the sub-representation $U_\rho^{(1)} := E_\rho U^{(1)}$ is equivalent to a finite number, say n_ρ , of copies of the irreducible “Wigner” representation of the universal covering of the Poincaré group with mass m_ρ and a certain spin $s_\rho \in \mathbb{R}$. Let us denote this representation by U_ρ^w . It acts on the Hilbert space $L^2(H_{m_\rho}^+, d\mu) \otimes \mathbb{C}^{n_\rho}$, which consists of momentum space “wave functions” $\psi : H_{m_\rho}^+ \rightarrow \mathbb{C}^{n_\rho}$ living on the mass shell $H_{m_\rho}^+ := \{p \in \mathbb{R}^3 \mid p \cdot p = m_\rho^2, p_0 > 0\}$ and having finite norm w.r.t. the scalar product

$$(\psi, \phi) = \int_{H_{m_\rho}^+} d\mu(p) (\psi(p), \phi(p))_{\mathbb{C}^{n_\rho}}.$$

The representation U_ρ^w acts in this space as (see e.g. [40])

$$(U_\rho^w(a, \tilde{\lambda})\psi)(p) = e^{is\Omega(\tilde{\lambda}, p)} e^{ia \cdot p} \psi(\lambda^{-1}p),$$

where λ is the Lorentz transformation onto which $\tilde{\lambda}$ projects, and $\Omega(\tilde{\lambda}, p) \in \mathbb{R}$ is the so-called Wigner rotation. To the representation U_ρ^w an anti-unitary operator $U_\rho^w(j)$ can be adjoined satisfying the representation properties

$$U_\rho^w(j)^2 = \mathbb{1} \quad \text{and} \quad U_\rho^w(j) U_\rho^w(\tilde{g}) U_\rho^w(j) = U_\rho^w(j\tilde{g}j) \quad (44)$$

for all $\tilde{g} \in \tilde{P}_+^\uparrow$. Namely, it is given by [40]

$$(U_\rho^w(j)\psi)(p) := \overline{\psi(-jp)},$$

where the overline denotes component-wise complex conjugation in \mathbb{C}^{n_ρ} . U_ρ^w is then a representation of the group \tilde{P}_+ which we identify with the semi-direct product of \tilde{P}_+^\uparrow and \mathbb{Z}_2 , the latter acting in the former via the unique lift [58] of the adjoint action of j on P_+^\uparrow . Let us denote the unitary intertwiner between the representations U_ρ^w and $U_\rho^{(1)}$ of \tilde{P}_+^\uparrow by W_ρ . In other words, W_ρ is an isometric isomorphism from $L^2(H_{m_\rho}^+, d\mu) \otimes \mathbb{C}^{n_\rho}$ onto $E_\rho^{(1)}\mathcal{H}$ satisfying

$$U_\rho^{(1)}(\tilde{g})W_\rho = W_\rho U_\rho^w(\tilde{g}) \quad (45)$$

for all $\tilde{g} \in \tilde{P}_+^\uparrow$. It is known that the masses [18], spins [42] and degeneracies n_ρ [42] coincide for ρ and $\bar{\rho}$. Hence the ranges of W_ρ^* and $W_{\bar{\rho}}^*$ coincide, and $C_\rho := W_{\bar{\rho}}W_\rho^*$ is a unitary operator from $E_\rho^{(1)}\mathcal{H}$ onto $E_{\bar{\rho}}^{(1)}\mathcal{H}$, representing ‘‘charge conjugation’’, which intertwines the representations $U_\rho^{(1)}$ and $U_{\bar{\rho}}^{(1)}$. Therefore the ‘‘CPT’’-operator

$$U^{(1)}(j) := \sum_{\rho \in \Sigma^{(1)}} U_\rho^{(1)}(j), \quad U_\rho^{(1)}(j) := W_{\bar{\rho}}U_\rho^w(j)W_\rho^* \equiv C_\rho W_\rho U_\rho^w(j)W_\rho^*, \quad (46)$$

not only conjugates the charge, but also represents the reflection j , namely satisfies

$$U^{(1)}(j)^2 = \mathbb{1}, \quad U^{(1)}(j)U^{(1)}(\tilde{g})U^{(1)}(j)^* = U^{(1)}(j\tilde{g}j). \quad (47)$$

We define now a closed anti-linear operator $S_{\text{geo}}^{(1)}$ in terms of the representation $U^{(1)}(\tilde{P}_+)$, as anticipated, by Eq. (43). Note that the group relation $j\tilde{\lambda}_1(t)j = \tilde{\lambda}_1(t)$ implies that $S_{\text{geo}}^{(1)}$ is an involution: it leaves its domain invariant and satisfies $(S_{\text{geo}}^{(1)})^2 \subset \mathbb{1}$.

Proposition 3.1 *There is a unitary operator D on $E^{(1)}\mathcal{H}$ commuting with the representation $U^{(1)}$ of \tilde{P}_+^\uparrow and with each E_ρ , such that*

$$S_{\text{geo}}^{(1)} = D S^{(1)}. \quad (48)$$

Proof. Let \mathcal{C}_1 denote the class of space-like cones which have apex at the origin, contain the positive x^1 -axis and are contained in the wedge W_1 , and have non-zero intersection with the time-zero hyper-surface, and let $\tilde{\mathcal{C}}_1$ be the set of (equivalence classes of) paths in \mathcal{K} of the form (C_0, W_1, C) with $C \in \mathcal{C}_1$. Let further $\mathcal{F}_{\iota, \rho}^\infty(\tilde{C})$ be the set of field operators $F \in \mathcal{F}_\iota(\tilde{C})$ with superselection channel $e = (\iota, \rho, \rho)$ and for which $\tilde{g} \mapsto \alpha_{\tilde{g}}(F)$ are smooth functions.

Lemma 3.2 ([11, 42]) *Let $\tilde{C} \in \tilde{\mathcal{C}}_1$ and $F \in \mathcal{F}_{\iota, \rho}^\infty(\tilde{C})$. Then, for fixed $p \in H_{m_\rho}^+$, the \mathbb{C}^{n_ρ} -valued function*

$$t \mapsto \psi(t, p) := (U_\rho^w(\tilde{\lambda}_1(t))W_\rho^*E_\rho^{(1)}F\Omega)(p) \quad (49)$$

extends to an analytic function in the strip $t \in \mathbb{R} + i(0, \pi)$, which is continuous and bounded on its closure. At $t = i\pi$, it has the boundary value

$$\overline{\psi(t, -jp)|_{t=i\pi}} = D_{\bar{\rho}}(W_{\bar{\rho}}^*E_{\bar{\rho}}^{(1)}F^\dagger\Omega)(p), \quad (50)$$

where $D_{\bar{\rho}}$ is an isometry in \mathbb{C}^{n_ρ} independent of \tilde{C} and F .¹⁶

¹⁶On the left hand side of Eq. (50), one first analytically continues into $t = i\pi$ and then conjugates component-wise in \mathbb{C}^{n_ρ} .

Proof. We show how this lemma follows from [42, Proposition 2], which in turn is based on the work of Buchholz and Epstein [11]. If $\tilde{C} \in \tilde{\mathcal{C}}_1$, then C contains a space-like cone C_1 of the special class used in [11] and [42, see Eq. (16)]. Defining $C_2 := -C_1$ and choosing a path \tilde{C}_2 ending at C_2 and satisfying $N(\tilde{C}_2, C_1) = -1$, all conditions on \tilde{C}_1 and \tilde{C}_2 used in [42] are satisfied. (In particular, the “dual” [42, Eq. (17)] of the difference cone $C_2 - C_1$ contains the negative x^1 -axis, as required in Eq. (36) of [42].) For $i \in \{1, 2\}$ we now pick n_ρ operators $F_{i,\beta} \in \mathcal{F}_{\iota,\rho}^\infty(\tilde{C}_i)$, $\beta = 1, \dots, n_\rho$, with $F_{1,1} := F$ of the lemma, such that the n_ρ vectors

$$(W_\rho^* E_\rho^{(1)} F_{i,\beta} \Omega)(p) \in \mathbb{C}^{n_\rho}$$

are linearly independent for all p in some open set.¹⁷ Then Proposition 2 in [42] asserts that there is an isometric matrix $D_{\bar{\rho}}$, independent of \tilde{C}_i and $F_{i,\beta}$, such that the assertion of our Lemma holds for all $F_{1,\beta}$. This completes the proof of the lemma. \square

We now reformulate the lemma in terms of the operators $S^{(1)}$ and $S_{\text{geo}}^{(1)}$. By the lemma, there is an operator A_ρ on $E_\rho^{(1)} \mathcal{H}$ with domain

$$\mathcal{D}_{0,\rho}^{(1)} := \text{span}_{\tilde{C} \in \tilde{\mathcal{C}}_1} E^{(1)} \mathcal{F}_{\iota,\rho}^\infty(\tilde{C}) \Omega$$

defined via

$$(W_\rho^* A_\rho \phi)(p) := (W_\rho^* U_\rho^{(1)}(\tilde{\lambda}_1(t)) \phi)(p)|_{t=i\pi}, \quad \phi \in \mathcal{D}_{0,\rho}^{(1)}. \quad (51)$$

Denoting by \hat{D}_ρ the multiplication operator with the matrix D_ρ ,

$$(W_\rho^* \hat{D}_\rho \phi)(p) := D_\rho (W_\rho^* \phi)(p),$$

Eq. (50) of the Lemma reads

$$U_\rho^{(1)}(j) A_\rho \subset \hat{D}_{\bar{\rho}} S^{(1)} E_\rho. \quad (52)$$

(Note that the domain $\mathcal{D}_{0,\rho}^{(1)}$ of A_ρ is contained in the domain of $S^{(1)}$.) We wish to identify A_ρ with $e^{-\pi K_\rho}$, where K_ρ is the generator of the one-parameter group $U_\rho^{(1)}(\tilde{\lambda}_1(t))$. First, relation (52) shows that A_ρ is closable since $S^{(1)}$ is. Now for ϕ in the dense domain of A_ρ^* , $\psi \in \mathcal{D}_{0,\rho}^{(1)}$ and $f \in C_0^\infty(\mathbb{R})$ one finds (by a calculation as in [39, proof of Lemma 11]):

$$(\phi, e^{-\pi K_\rho} f(K_\rho) \psi) = (A_\rho^* \phi, f(K_\rho) \psi).$$

This implies that the span of vectors of the form $f(K_\rho) \psi$ as above, $\mathcal{D}_\rho^{(1)}$, is in the domain of the closure A_ρ^{**} of A_ρ , and that on $\mathcal{D}_\rho^{(1)}$ the closure A_ρ^{**} coincides with $e^{-\pi K_\rho}$. But $\mathcal{D}_\rho^{(1)}$ is invariant under the unitary one-parameter group $U_\rho^{(1)}(\tilde{\lambda}_1(t))$, because for each t there is some $\tilde{C}_t \in \tilde{\mathcal{C}}_1$ such that $\lambda_1(t)\tilde{C} \subset \tilde{C}_t$. By standard arguments, the closure of A_ρ then coincides with the operator $e^{-\pi K_\rho}$. In particular, $\mathcal{D}_{0,\rho}^{(1)}$ is a core for $e^{-\pi K_\rho}$, $\rho \in \Sigma^{(1)}$, hence $\bigoplus_\rho \mathcal{D}_{0,\rho}^{(1)}$ is a core for $S_{\text{geo}}^{(1)}$. Therefore, relation (52) implies

$$S_{\text{geo}}^{(1)} \subset D S^{(1)}, \quad (53)$$

¹⁷If $E_\rho^{(1)} F \Omega$ is non-zero, then this is possible due to the Reeh-Schlieder property.

where $D := \bigoplus_\rho \hat{D}_\rho$. By construction, D commutes with E_ρ and with the representation $U^{(1)}$, as claimed in the Proposition. It remains to show the opposite inclusion “ \supset ” in Eq. (53). To this end, we refer to the opposite wedge $W'_1 = r(\pi) W_1$. Let

$$\begin{aligned}\hat{S} &:= U(\tilde{r}(-\pi)) S U(\tilde{r}(\pi)), \\ \hat{S}^{(1)} &:= U^{(1)}(\tilde{r}(-\pi)) S^{(1)} U^{(1)}(\tilde{r}(\pi)) \equiv \hat{S} E^{(1)}, \\ \hat{S}_{\text{geo}}^{(1)} &:= U^{(1)}(\tilde{r}(-\pi)) S_{\text{geo}}^{(1)} U^{(1)}(\tilde{r}(\pi)).\end{aligned}$$

We claim that the following sequence of relations holds true:

$$S^{(1)} \subset \omega \chi (\hat{S}^{(1)})^* \subset \omega D^* (\hat{S}_{\text{geo}}^{(1)})^* = \omega U^{(1)}(\tilde{r}(-2\pi)) D^* S_{\text{geo}}^{(1)}. \quad (54)$$

Here, $\omega := \sum_\rho \omega_\rho E_\rho$ where ω_ρ are the statistics phases, and χ is the operator defined in Eq. (36). To see the first inclusion, note that \hat{S} is the closure of the operator $F\Omega \mapsto F^\dagger\Omega$, $F \in \mathcal{F}_\ell(\tilde{r}(-\pi)\tilde{W}_1)$. Therefore Eq.s (26) and (21) imply that for $F_1 \in \mathcal{F}_\ell(\tilde{W}_1)$ and $F_2 \in \mathcal{F}_\ell(\tilde{r}(-\pi)\tilde{W}_1)$, there holds

$$(\omega \chi^* \hat{S} F_2^\dagger \Omega, F_1 \Omega) = (S F_1 \Omega, F_2^\dagger \Omega).$$

This implies $S \subset (\omega \chi^* \hat{S})^* \equiv \hat{S}^* \chi \omega^*$, which coincides with $\omega \chi \hat{S}^*$ because $\chi_{\bar{\rho}} = \overline{\chi_\rho}$ and $\omega_{\bar{\rho}} = \omega_\rho$, while \hat{S}^* is anti-linear and maps $E_\rho \mathcal{H}$ into $E_{\bar{\rho}} \mathcal{H}$. Since S commutes with $E^{(1)}$ due to Borchers' commutation relations, this implies the first inclusion in (54). To show the second inclusion, we first note that relations (53) and (36) imply $(D^* S_{\text{geo}}^{(1)})^2 \subset (S^{(1)})^2 \subset \chi$. Since $\hat{S}_{\text{geo}}^{(1)} = (\hat{S}_{\text{geo}}^{(1)})^{-1}$, this yields

$$D^* S_{\text{geo}}^{(1)} = \chi S_{\text{geo}}^{(1)} D \equiv S_{\text{geo}}^{(1)} D \chi$$

(we have equality here instead of \subset , since D leaves the domain of $S_{\text{geo}}^{(1)}$ invariant), and the same relation holds for $\hat{S}_{\text{geo}}^{(1)}$. Therefore, the adjoint of Eq. (53) yields

$$(\hat{S}^{(1)})^* \subset (D^* \hat{S}_{\text{geo}}^{(1)})^* \equiv \chi^* D^* (\hat{S}_{\text{geo}}^{(1)})^*,$$

which implies the second inclusion in (54). As to the last equality in Eq. (54), note that the group relations $j\tilde{r}(\omega)j = \tilde{r}(-\omega)$, $\tilde{r}(\pi)\tilde{\lambda}_1(t)\tilde{r}(-\pi) = \tilde{\lambda}_1(-t)$ and $j\tilde{\lambda}_1(t)j = \tilde{\lambda}_1(t)$ imply that $(\hat{S}_{\text{geo}}^{(1)})^*$ coincides with $U^{(1)}(\tilde{r}(-2\pi)) S_{\text{geo}}^{(1)}$. This completes the proof of the sequence of relations (54). Using relation (53) then yields

$$S^{(1)} \subset \omega U^{(1)}(\tilde{r}(-2\pi)) D^* S_{\text{geo}}^{(1)} \subset \omega U^{(1)}(\tilde{r}(-2\pi)) S^{(1)}.$$

This implies firstly that $\omega = U^{(1)}(\tilde{r}(2\pi))$ (which is the spin-statistics theorem) and secondly that $S^{(1)} \subset D^* S_{\text{geo}}^{(1)}$, and completes the proof of the Proposition. \square

Note that the proof shows that, although we use results from [42], the spin-statistics connection needs not be assumed but rather follows.

By uniqueness of the polar decomposition, equation (48) of the proposition implies the equations

$$\Delta^{\frac{1}{2}} E^{(1)} = e^{-\pi K} E^{(1)}, \quad D J E^{(1)} = U^{(1)}(j) E^{(1)}.$$

Since the unitary D commutes with $U^{(1)}(\tilde{P}_+^\dagger)$, the above equations and Eq. (47) imply the single particle versions of the Bisognano-Wichmann and CPT theorems:

Corollary 3.3 *Let the Assumption A1 of Section 1 hold. Then*

i) Modular Covariance holds on the single particle space:

$$\Delta^{it} E^{(1)} = U(\lambda_1(-2\pi t)) E^{(1)}. \quad (55)$$

ii) $J E^{(1)}$ is a “CPT operator” on $E^{(1)}\mathcal{H}$, namely, for all $\tilde{g} \in \tilde{P}_+^\uparrow$ holds

$$J U(\tilde{g}) J^{-1} E^{(1)} = U(j \tilde{g} j) E^{(1)}. \quad (56)$$

Note that J is not an involution, but rather satisfies $J^2 = \chi$, c.f. Eq. (35).

4 Modular Covariance on the Space of Scattering States

We shall now show that modular covariance, which we have established on the single particle space, extends to multi-particle states via (Haag-Ruelle) scattering theory. Haag-Ruelle scattering theory, as developed in [32, 33], associates a multi-particle state to n single particle vectors which are created from the vacuum by quasi-local field operators. It has been adapted, within the field bundle formulation, to the setting of algebraic quantum field theory in [14], to theories with topological charges (i.e., charges localized in space-like cones) in [12], and to theories with braid group statistics in [21]. Since we are not aware of an exposition of scattering theory within the *reduced* field bundle framework, we shall give a brief such exposition here. For $\rho \in \Sigma^{(1)}$ and $\tilde{C} \in \tilde{\mathcal{C}}$, let $F = F(e, A)$ be a field operator in $\mathcal{F}(\tilde{C})$ carrying charge ρ , which produces from the vacuum a single particle vector with non-zero probability in the sense that it satisfies $E^{(1)} F_\ell \Omega \neq 0$. Here we have written

$$\begin{aligned} F_\ell &:= F(e_\ell, A) & \text{if} & \quad F = F(e, A), \\ e_\ell &:= (\ell, \rho, \rho) & \text{if} & \quad e = (\rho_s, \rho, \rho_r, i). \end{aligned}$$

The mentioned quasi-local creation operator is constructed from F as follows. Let $f \in \mathcal{S}(\mathbb{R}^3)$ be a Schwartz function whose Fourier transform \tilde{f} has compact support contained in the open forward light cone V_+ and intersects the energy momentum spectrum of the sector ρ only in the mass shell $H_{m_\rho}^+$. Recall that the latter is assumed to be isolated from the rest of the energy momentum spectrum in the sector $E_\rho \mathcal{H}$. Let now

$$\begin{aligned} f_t(x) &:= (2\pi)^{-2} \int d^3 p e^{i(p_0 - \omega_\rho(\mathbf{p}))t} e^{-ip \cdot x} \tilde{f}(p), \\ F(f_t) &:= \int d^3 x f_t(x) \alpha_x(F), \end{aligned}$$

where $\omega_\rho(\mathbf{p}) := (\mathbf{p}^2 + m_\rho^2)^{1/2}$. For large $|t|$, the operator $F(f_t)$ is essentially localized in $\tilde{C} + t V_\rho(f)$, where $V_\rho(f)$ is the velocity support of f ,

$$V_\rho(f) := \left\{ \left(1, \frac{\mathbf{p}}{\omega_\rho(\mathbf{p})} \right), p = (p^0, \mathbf{p}) \in \text{supp } \tilde{f} \right\}. \quad (57)$$

Namely, for any $\varepsilon > 0$, $F(f_t)$ can be approximated by an operator $F_t^\varepsilon \in \mathcal{F}(\tilde{C} + t V^\varepsilon)$, where V^ε is an ε -neighborhood of $V_\rho(f)$, in the sense that $\|F_t^\varepsilon - F(f_t)\|$ is of fast decrease in t [6, 32]. Further, $F_\ell(f_t)$ creates from the vacuum a single particle vector

$$F_\ell(f_t) \Omega = \tilde{f}(P) F_\ell \Omega \in E_\rho^{(1)} \mathcal{H}, \quad (58)$$

which is independent of t , and whose velocity support is contained in that of f . (Here the velocity support of a single particle vector is defined as in Eq. (57), with the spectral support of ψ taking the role of $\text{supp } \tilde{f}$.) To construct an outgoing scattering state from n single particle vectors, pick n localization regions \tilde{C}_i , $i = 1, \dots, n$ and compact sets V_i in velocity space, such that for suitable open neighborhoods $V_i^\varepsilon \subset \mathbb{R}^3$ the regions $\tilde{C}_i + t V_i^\varepsilon$ are mutually space-like separated for large t . Next, choose $F_i \in \mathcal{F}(\tilde{C}_i)$ with respective superselection channels e_i suitably chosen such that the vector $F_n \cdots F_1 \Omega$ does not vanish from ‘‘algebraic’’ reasons, i.e. satisfying $s(e_1) = \iota$ and $s(e_i) = r(e_{i-1})$, $i = 2, \dots, n$. Choose further Schwartz functions f_i as above with $V_{\rho_i}(f_i) \subset V_i$. Then the standard lemma of scattering theory, in the present context, asserts the following:

Lemma 4.1 *The limit*

$$\lim_{t \rightarrow \infty} F_n(f_{n,t}) \cdots F_1(f_{1,t}) \Omega =: (\psi_n \times \cdots \times \psi_1)^{\text{out}} \quad (59)$$

exists and depends only on the single particle vectors $\psi_i := (F_i)_\iota(f_{i,t}) \Omega$, on the localization regions \tilde{C}_i and on the superselection channels e_i .¹⁸ The limit vector is approached faster than any inverse power of t , and depends continuously on the single particle vectors ψ_i . Further, there holds

$$(\psi_n \times \cdots \times \psi_1)^{\text{out}} = \lim_{t \rightarrow \infty} F_n(f_{n,t}) (\psi_{n-1} \times \cdots \times \psi_1)^{\text{out}}. \quad (60)$$

Proof. The space-like commutation relations (23) imply that, for $k \in \{2, \dots, n\}$,

$$F_k(f_{k,t}) \cdots F_1(f_{1,t}) \Omega \simeq \sum_{\rho_2, \dots, \rho_{k-1}} R_{k-1} \cdots R_1 F_{k-1}(f_{k-1,t}) \cdots F_1(f_{1,t}) \hat{F}_k(f_{k,t}) \Omega \quad (61)$$

up to terms which are of fast decrease in t . Here, $R_i = R(\rho_i, e_k^{(i)}, e_i, n_k^{(i)})$, where $e_k^{(i)}$ has the same charge as e_k , the same source as e_i , and range ρ_i , and $n_k^{(i)} = N(\tilde{C}_k, \tilde{C}_i)$. The sum goes over all ρ_i contained in the product $c(e_k)s(e_i)$, $i = 1, \dots, k-1$,¹⁹ plus the internal indices of the $e_k^{(i)}$. Further, \hat{F}_k is the operator arising from F_k by substituting its superselection channel e_k for $e_k^{(1)} \equiv (\iota, \rho_1, \rho_1)$ with $\rho_1 = c(e_k)$. From here the proof goes through as in [14]: Differentiating $F_n(f_{n,t}) \cdots F_1(f_{1,t}) \Omega$ with respect to t yields a sum of terms of the form $F_n(f_{n,t}) \cdots (\frac{d}{dt} F_k(f_{k,t})) \cdots F_1(f_{1,t}) \Omega$. Now $\frac{d}{dt} F_k(f_{k,t})$ is of the same form as $F_k(f_{k,t})$, hence can be permuted to the right by Eq. (61) up to fast decreasing terms, where it annihilates the vacuum due to Eq. (58). This shows the fast convergence in (59). Let now G_k be a field operator with the same localization \tilde{C}_k and superselection channel e_k as F_k and such that $(G_k)_\iota(g_k)$ creates the same single particle vector ψ_k from the vacuum as $(F_k)_\iota(f_k)$. Then Eq. (61) still holds, with the same numbers R_i , when $F_k(f_{k,t})$ is replaced by $G_k(g_{k,t})$. This implies that the scattering state (59) only depends on the single particle states, localization regions and superselection channels. The continuous dependence on the single particle vectors follows from the local tensor product structure derived in [21, Thm. 3.2]. Eq. (60) follows as in [32] from the facts that $F_{n-1}(f_{n-1,t}) \cdots F_1(f_t) \Omega$ converges rapidly to $(\psi_{n-1} \times \cdots \times \psi_1)^{\text{out}}$, while $\|F_n(f_{n,t})\|$ increases at most like $|t|^3$. \square

¹⁸We omit the dependence on \tilde{C}_i and e_i in our notation.

¹⁹ ρ_1 does not appear in the sum because $c(e_k)s(e_1) \equiv c(e_k)$ contains no other representation than $\rho_1 \equiv c(e_k)$. In particular there is no sum if $k = 2$.

Let us denote by $\mathcal{H}^{(n)}$, $n \geq 2$, the closed span of outgoing n -particle scattering states as in the lemma, and by $\mathcal{H}^{(\text{out})}$ the span of all (outgoing) particle states:

$$\mathcal{H}^{(\text{out})} := \bigoplus_{n \in \mathbb{N}_0} \mathcal{H}^{(n)}.$$

Here $\mathcal{H}^{(0)}$ is understood to be the span of the vacuum vector Ω and $\mathcal{H}^{(1)} := E^{(1)}\mathcal{H}$. *Asymptotic completeness* (our Assumption A2) means that $\mathcal{H}^{(\text{out})}$ coincides with \mathcal{H} . Our main result is now

Theorem 4.2 (Covariance of the Pseudo-Modular Groups.) *Let the Assumptions A1 and A2 of Section 1 hold. Then the field algebra satisfies covariance of the pseudo-modular groups,*

$$\Delta^{it} = U(\lambda_1(-2\pi t)). \quad (62)$$

(Of course, this is equivalent to $\Delta_\rho^{it} = U_\rho(\lambda_1(-2\pi t))$, $\rho \in \Sigma$.)

Proof. On the single particle space, the claim is our Corollary 3.3. For the scattering states, one proves Eq. (62) by induction over the particle number. Here, we can literally take over the proof from [39, Prop. 7] due to the last lemma and the results of Section 2, in particular Borchers' commutation relations (32) and invariance of the wedge field algebra under the pseudo-modular group (38). Asymptotic completeness then implies that Eq. (62) holds on the entire Hilbert space \mathcal{H} . \square

Since for observables the pseudo-adjoint F^\dagger coincides with the operator adjoint, this theorem asserts in particular modular covariance of the observables and implies, as mentioned in the introduction, the CPT theorem on the observable level. (Of course for these properties to hold true, asymptotic completeness on the *vacuum* sector is a sufficient condition, namely $\mathcal{H}^{(\text{out})} \cap \mathcal{H}_\iota = \mathcal{H}_\iota$.)

5 The CPT Theorem

We now prove that the pseudo-modular conjugation J is a CPT operator on the level of the field algebra, namely satisfies Eq.s (4), (5) and (6). As mentioned in the introduction, Guido and Longo have shown [29] that modular covariance of the observables, which we have established in Theorem 4.2, implies that the corresponding modular conjugation J_0 is a “PT” operator on the observable level, namely satisfies Eq. (4) on the vacuum Hilbert space and Eq. (5) with $\mathcal{A}(C)$ instead of $\mathcal{F}(C)$. As a consequence, the anti-isomorphism $\alpha_j : \mathcal{A}(W_1) \rightarrow \mathcal{A}(W_1')$ implemented by J_0 , cf. Eq. (31), extends to an anti-automorphism of the entire universal algebra \mathcal{A} (still implemented by J_0). In particular, we have

Corollary 5.1 ([29]) *The modular conjugation J_0 of the observable algebra associated with the wedge W_1 implements an anti-automorphism α_j of the universal algebra \mathcal{A} ,*

$$\text{Ad}J_0 \circ \pi_0 = \pi_0 \circ \alpha_j, \quad (63)$$

which has the representation properties $\alpha_j \alpha_g \alpha_j = \alpha_{ggj}$, $\alpha_j^2 = \iota$, and acts geometrically correctly:

$$\alpha_j : \mathcal{A}(I) \rightarrow \mathcal{A}(jI), \quad I \in \mathcal{K}. \quad (64)$$

Guido and Longo also show that α_j intertwines any localized morphism $\rho \in \Sigma$ with its conjugate $\bar{\rho}$ up to equivalence:

$$\alpha_j \circ \rho \circ \alpha_j \simeq \bar{\rho}. \quad (65)$$

(In fact, we exhibit an intertwiner establishing this equivalence in Lemma 5.5 below.) In order to make the CPT theorem more explicit on the level of the field algebra, we begin with establishing the representation and implementation properties of the relative modular conjugations J_ρ .

Proposition 5.2 *The relative modular conjugations J_ρ , $\rho \in \Sigma$, represent the reflection j in the direct sum representation $U_\rho \oplus U_{\bar{\rho}}$, and implement α_j in the direct sum representation $\pi_0 \rho \oplus \pi_0 \bar{\rho}$ of \mathcal{A} . Namely, there holds*

$$J_\rho U_\rho(\tilde{g}) J_\rho^{-1} = U_{\bar{\rho}}(j\tilde{g}j), \quad \tilde{g} \in \tilde{P}_+^\uparrow, \quad (66)$$

$$\text{Ad}J_\rho \circ \pi_0 \rho = \pi_0 \bar{\rho} \circ \alpha_j \quad \text{on } \mathcal{A}. \quad (67)$$

Of course, Eq. (66) implies that also the pseudo-modular conjugation J represents the reflection, $JU(\tilde{g})J^{-1} = U(j\tilde{g}j)$.

Proof. Eq. (66) corresponds to Prop. 2.8 of Guido and Longo's article [29], whose proof uses Borchers' commutation relations, commutation of the modular unitary group with the modular conjugation, and an assertion [29, Thm. 1.1] about unitary representations of the universal covering group of $SL(2, \mathbb{R})$. But the commutation relations also hold in the present setting, c.f. Eq.s (32) through (35), and the universal covering group of $SL(2, \mathbb{R})$ is just the homogeneous part of our \tilde{P}_+^\uparrow , hence the proof of Guido and Longo goes through in the present case. The implementing property (67) has been shown in [41, Prop. 1] to hold on $\mathcal{A}(W_1)$. By Eq. (66) it extends to all wedges $W = gW_1$. Considering a space-like cone C , we note that C coincides with the intersection of the wedge regions containing C . Then Haag duality for wedges and for space-like cones implies that

$$\mathcal{A}(C) = \bigcap_{W \supset C} \mathcal{A}(W).$$

(This can be seen by the arguments in the proof of Cor. 3.5 in [10].) This implies that Eq. (67) also holds for space-like cones, and further, again by Haag duality, for their causal complements. Thus Eq. (67) holds for all $I \in \mathcal{K}$, and the proof is complete. \square

Before calculating the adjoint action of J on the fields, we discuss the item of geometrical correctness (5). In order to formulate it, one needs an action of j on the set $\tilde{\mathcal{K}}$ of paths of space-like cones, such that $j \cdot \tilde{I}$ is a path which starts at C_0 and ends at jI . Since j cannot be continuously transformed to the identity transformation, such action must be of the form

$$j \cdot (C_0, I_1, \dots, I_n) = (C_0, I, jC_0, jI_1, \dots, jI_n),$$

where $I \in \mathcal{K}$ contains both C_0 and jC_0 . Now our requirement that C_0 be contained in W_1 implies that C_0 and jC_0 are causally separated. Hence there are two topologically distinct regions I^\pm containing $C_0 \cup jC_0$, and the action of j on $\tilde{\mathcal{K}}$ depends on this choice. The action would be canonical, however, if the reference cone C_0 (which enters in the

definition of $\tilde{\mathcal{K}}$) were invariant under j .²⁰ Namely, then j would act canonically as

$$j \cdot (C_0, I_1, \dots, I_n) := (jC_0, jI_1, \dots, jI_n). \quad (68)$$

We therefore wish to go over to such a reference cone. To this end we first convince ourselves that all relevant structure of the field algebra is preserved under such change. More precisely, we show that two field algebras constructed as in Section 1 from the same observables and sectors, but with different reference cones, are isomorphic in all structural elements mentioned in Section 1:

Lemma 5.3 *Let $\hat{\Sigma}$ be a collection of morphisms localized in a reference cone \hat{C}_0 (instead of C_0), one per equivalence class, and let $\hat{\mathcal{F}}$, $\hat{\mathcal{K}}$, $\hat{\mathcal{F}}(\tilde{I})$ for $\tilde{I} \in \hat{\mathcal{K}}$, $\hat{\mathcal{H}}$ and $\hat{U}(\tilde{g})$ be defined as in Section 1 with Σ replaced by $\hat{\Sigma}$. Then there is a bijection $\tilde{I} \mapsto \hat{\tilde{I}}$ from $\tilde{\mathcal{K}}$ onto $\hat{\mathcal{K}}$ and an isometric isomorphism $W : \mathcal{H} \rightarrow \hat{\mathcal{H}}$ which implements a unitary equivalence $\mathcal{F} \cong \hat{\mathcal{F}}$ preserving the respective notions of localization and pseudo-adjoints, and intertwining the representations U and \hat{U} as well as the vacua. In formulas:*

$$\text{Ad}W : \mathcal{F}(\tilde{I}) \rightarrow \hat{\mathcal{F}}(\hat{\tilde{I}}), \quad \tilde{I} \in \tilde{\mathcal{K}}, \quad (69)$$

$$WF^\dagger W^* = (WFW^*)^\dagger, \quad F \in \mathcal{F}_\iota, \quad (70)$$

$$WU(\tilde{g})W^* = \hat{U}(\tilde{g}), \quad \tilde{g} \in \tilde{P}_+^\dagger, \quad (71)$$

$$W\Omega = \hat{\Omega}. \quad (72)$$

Proof. We choose $I_0 \in \mathcal{K}$ which contains both C_0 and \hat{C}_0 , and pick for each $\rho \in \Sigma$ with corresponding $\hat{\rho} \in \hat{\Sigma}$ a unitary intertwiner $W_\rho \in \mathcal{A}(I_0)$ such that $\hat{\rho} = \text{Ad}W_\rho \circ \rho$. Note that $\hat{\rho}$ coincides with the conjugate of $\hat{\rho}$, since Σ and $\hat{\Sigma}$ contain exactly one morphism per class. As intertwiner from ι to $\hat{\rho}\hat{\rho}$ we choose

$$R_{\hat{\rho}} := W_{\bar{\rho}} \bar{\rho}(W_\rho) R_\rho \equiv \hat{\rho}(W_\rho) W_{\bar{\rho}} R_\rho.$$

For $e = (\rho_s, \rho_c, \rho_r, i)$ let $\hat{e} := (\hat{\rho}_s, \hat{\rho}_c, \hat{\rho}_r, i)$ and

$$T_{\hat{e}} := (W_{\rho_s} \times W_{\rho_c}) T_e W_{\rho_r}^* \equiv W_{\rho_s} \rho_s(W_{\rho_c}) T_e W_{\rho_r}^* = \hat{\rho}_s(W_{\rho_c}) W_{\rho_s} T_e W_{\rho_r}^*.$$

If i runs through $1, \dots, \dim \text{Int}(\rho_s \rho_c | \rho_r)$, then the $T_{\hat{e}}$ are an orthonormal basis of $\text{Int}(\hat{\rho}_s \hat{\rho}_c | \hat{\rho}_r)$. Let finally $W : \mathcal{H} \rightarrow \hat{\mathcal{H}}$ be defined by $W(\rho, \psi) := (\hat{\rho}, \pi_0(W_\rho)\psi)$. Then there holds $WF(e, A)W^* = \hat{F}(\hat{e}, W_\rho A)$, where ρ is the charge of e . Defining a bijection $\tilde{\mathcal{K}} \rightarrow \hat{\mathcal{K}}$ by

$$(C_0, I_1, \dots, I_n) = \tilde{I} \mapsto \hat{\tilde{I}} = (\hat{C}_0, I_0, C_0, I_1, \dots, I_n),$$

one also checks that $F(e, A) \in \mathcal{F}(\tilde{I})$ if and only if $\hat{F}(\hat{e}, A) \in \hat{\mathcal{F}}(\hat{\tilde{I}})$. Eq. (70) is easily verified, and the intertwiner relation (71) follows from [15, Lem. 2.2]. \square

The lemma implies of course that the pseudo-modular objects of $\hat{\mathcal{F}}$ have the same algebraic relations among themselves and with the representation $\hat{U}(\tilde{P}_+^\dagger)$ as those of \mathcal{F} . Namely, it implies that the new pseudo-Tomita operator \hat{S} defined via

$$\hat{S} \hat{F} \hat{\Omega} := \hat{F}^\dagger \hat{\Omega}, \quad \hat{F} \in \hat{\mathcal{F}}(\hat{W}_1),$$

satisfies $\hat{S} = WSW^*$, and we have the following

²⁰Again, there are two topologically distinct possibilities for the choice of C_0 : It must contain either the positive or of the negative x^2 -direction. The difference between the two choices shows up, in our context, only in the action of the twist operator for Anyons, see Eq. (88). So far, we leave the choice of C_0 unspecified.

Corollary 5.4 *All commutation relations between the relative modular objects and the Poincaré transformations, namely Eq.s (34, 35, 62, 66), as well as the implementation property (67), also hold with $\rho, \Delta_\rho, J_\rho, U_\rho$ replaced by $\hat{\rho}, \hat{\Delta}_{\hat{\rho}}, \hat{J}_{\hat{\rho}}, \hat{U}_{\hat{\rho}}$, respectively.*

Here, $\hat{J}_{\hat{\rho}}$ and $\hat{\Delta}_{\hat{\rho}}$ are defined from $\hat{S} \equiv \hat{J} \hat{\Delta}^{1/2}$ as in Eq.s (27), (37).

Proof. Only the implementation property (67) of the new conjugation $\hat{J}_{\hat{\rho}}$ remains to be shown. But this property follows from the fact that W_ρ intertwines the morphisms $\hat{\rho}$ and ρ by construction, c.f. the proof of Lemma 5.3. \square

Due to this result, we may from now on assume that the reference cone C_0 in which all morphisms $\rho \in \Sigma$ are localized is invariant under j , $jC_0 = C_0$, and that j acts on $\tilde{\mathcal{K}}$ in a canonical way as in Eq. (68). We now wish to calculate the action of J on the field algebra, and to this end introduce “cocycle” type unitaries V_ρ which implement the equivalence (65) of $\alpha_j \rho \alpha_j$ and $\bar{\rho}$:

Lemma 5.5 *The unitary operator $J_\rho J_0$ is in $\mathcal{A}_0(C_0)$. Its pre-image under the (faithful) restriction of π_0 to $\mathcal{A}(C_0)$ is an intertwiner from $\alpha_j \rho \alpha_j$ to $\bar{\rho}$. In other words, the unitary*

$$V_\rho := \pi_0^{-1}(J_\rho J_0) \quad (73)$$

is in $\mathcal{A}(C_0)$ and satisfies

$$\text{Ad}V_\rho \circ \alpha_j \rho \alpha_j = \bar{\rho}. \quad (74)$$

Proof. The implementation properties (63) and (67) imply that

$$\text{Ad}(J_\rho J_0) \circ \pi_0 \alpha_j \rho \alpha_j = \pi_0 \bar{\rho}.$$

The morphisms on the left and right hand sides are localized in C_0 and jC_0 , respectively. Hence by Haag duality, $J_\rho J_0$ is in $\mathcal{A}_0(I) = \pi_0 \mathcal{A}(I)$, whenever I contains $C_0 \cup jC_0$. The rest of the proof is straightforward. \square

Our explicit formula for the adjoint action of J below uses the unitaries V_ρ and relies on the following observation. Recall that for each superselection channel $e = (\rho_s, \rho_c, \rho_r, i)$ we have chosen an intertwiner T_e in $\text{Int}(\rho_s \rho_c | \rho_r)$. One easily verifies that

$$\bar{T}_e := (V_{\rho_s} \times V_{\rho_c}) \alpha_j(T_e) V_{\rho_r}^* \equiv \bar{\rho}_s(V_{\rho_c}) V_{\rho_s} \alpha_j(T_e) V_{\rho_r}^*$$

is an intertwiner in $\text{Int}(\bar{\rho}_s \bar{\rho}_c | \bar{\rho}_r)$, and can therefore be expanded in terms of the basis $\{\bar{T}_{\bar{e}}\}$, where \bar{e} is of type $(\bar{\rho}_s, \bar{\rho}_c, \bar{\rho}_r)$. Therefore,

$$\bar{T}_e = \sum_{\bar{e}} c_{\bar{e},e} \bar{T}_{\bar{e}}, \quad \text{with} \quad c_{\bar{e},e} \mathbb{1} := \bar{T}_{\bar{e}}^* \bar{T}_e, \quad (75)$$

where the sum goes over all superselection channels \bar{e} of type $(\bar{\rho}_s, \bar{\rho}_c, \bar{\rho}_r)$. We are now prepared for the CPT theorem.

Theorem 5.6 (CPT Theorem.) *The pseudo-modular conjugation J is a CPT operator in the sense of Eq.s (4), (5) and (6). Namely, it represents the reflection j in a geometrically correct way:*

$$J U(\tilde{g}) J^{-1} = U(j \tilde{g} j), \quad \tilde{g} \in \tilde{P}_+^\dagger, \quad (76)$$

$$\text{Ad}J : \mathcal{F}(\tilde{I}) \rightarrow \mathcal{F}(j \cdot \tilde{I}), \quad \tilde{I} \in \tilde{\mathcal{K}}, \quad (77)$$

and conjugates charges. More explicitly, $\text{Ad}J$ is given by

$$J F(e, A) J^{-1} = \sum_{\bar{e}} \bar{c}_{\bar{e}, e} F(\bar{e}, V_{\rho_e} \alpha_j(A)), \quad (78)$$

where the sum goes over all superselection channels \bar{e} of type $(\bar{\rho}_s, \bar{\rho}_c, \bar{\rho}_r)$ if e is of type (ρ_s, ρ_c, ρ_r) .

It is interesting to note the resemblance of Eq.s (73) and (78) with Eq.s (19) and (18), respectively.

Proof. The representation property (76) has been shown in Proposition 5.2. To prove Eq. (78), let e be of type (ρ_s, ρ_c, ρ_r) . We have

$$\begin{aligned} J F(e, A) J^{-1}(\bar{\rho}_s, \psi) &= (\bar{\rho}_r, \pi_0(V_{\rho_r} \alpha_j(T_e^*) V_{\rho_s}^* \bar{\rho}_s \alpha_j(A)) \psi) \\ &= \sum_{\bar{e}} \bar{c}_{\bar{e}, e} (\bar{\rho}_r, \pi_0(T_{\bar{e}}^* \bar{\rho}_s (V_{\rho_c} \alpha_j(A)) \psi) \\ &= \sum_{\bar{e}} \bar{c}_{\bar{e}, e} F(\bar{e}, V_{\rho_c} \alpha_j(A)) (\bar{\rho}_s, \psi). \end{aligned}$$

In the second equality we have used that $V_{\rho_r} \alpha_j(T_e^*) V_{\rho_s}^* = (\bar{T}_e)^* \bar{\rho}_s (V_{\rho_c})$ by definition of \bar{T}_e , and substituted \bar{T}_e as in Eq. (75). This proves Eq. (78), and shows that $\text{Ad}J$ is an anti-automorphism of the field algebra. In order to show that it acts geometrically correctly, let $\tilde{I} = (I_0 = C_0, I_1, \dots, I_n = I) \in \tilde{\mathcal{K}}$, let $U = U_n \cdots U_1$ be a charge transporter for ρ along \tilde{I} , and let $F(e, A)$ be in $\mathcal{F}(\tilde{I})$ with $c(e) = \rho$. We define

$$\bar{U}_k := \alpha_j(U_k), \quad k = 1, \dots, n.$$

Then $\bar{U} := \bar{U}_n \cdots \bar{U}_1 V_{\rho}^*$ is a charge transporter for $\bar{\rho}$ along $j \cdot \tilde{I}$, since

$$\text{Ad}(\bar{U}_k \cdots \bar{U}_1 V_{\rho}^*) \circ \bar{\rho} \equiv \alpha_j \circ \text{Ad}(U_k \cdots U_1) \circ \rho \circ \alpha_j$$

is localized in jI_k , $k = 1, \dots, n$. Further, $\bar{U} V_{\rho} \alpha_j(A) \equiv \alpha_j(UA)$ is in $\mathcal{A}(jI)$ by Eq. (64), since by hypothesis $UA \in \mathcal{A}(I)$. Hence $F(\bar{e}, V_{\rho_c} \alpha_j(A))$ is in $\mathcal{F}(j \cdot \tilde{I})$, and the proof is complete. \square

6 Anyons

We now consider the case of Anyons, i.e., when all sectors correspond to automorphisms of the observable algebra. In this case, one can construct a field algebra \mathcal{F}_a [38, 47, 53] in the sense of the WWW scenario, in particular the vacuum vector is cyclic and separating for the local field algebras. Thus, the Tomita operator associated with $\mathcal{F}_a(\tilde{W}_1)$ and the vacuum is well-defined and one may ask whether there holds modular covariance. We show that this is indeed the case. To this end, we exhibit \mathcal{F}_a as a sub-algebra of the reduced field bundle \mathcal{F} , and show that the pseudo-modular operator of $\mathcal{F}(\tilde{W}_1)$ coincides with the (genuine) Tomita operator of $\mathcal{F}_a(\tilde{W}_1)$. Then modular covariance is implied by the results from the previous sections, and the CPT theorem follows in analogy with the permutation group statistics case.

We assume for simplicity that the set $\Sigma^{(1)}$ of elementary charges consists of only one automorphism, localized in $C_0 = jC_0$. (All results are easily transferred to the case

of finitely many elementary Abelian charges, i.e., $\Sigma^{(1)}$ finite.) Σ has then the structure of an Abelian group with one generator, i.e., \mathbb{Z}_N if there is a natural number N such that the N -fold product of the generating automorphism is equivalent to the identity, and \mathbb{Z} otherwise. In the former case there may or may not be a representant whose N -fold product coincides with the identity, as Rehren has pointed out [47]. We wish to exclude the case with obstruction for simplicity, and therefore make the following

Assumption A3 *The set of relevant sectors Σ is generated by one automorphism, γ . If there is a natural number N such that $\gamma^N := \gamma \circ \dots \circ \gamma$ is equivalent to the identity ι , then γ can be chosen such that $\gamma^N = \iota$.*

6.1 The Field Algebra for Anyons.

In the assumed absence of an obstruction, there is a field algebra \mathcal{F}_a in the sense of the WWW scenario as mentioned in the introduction: The dual group of Σ , in our case $U(1)$ or \mathbb{Z}_N , acts as a global gauge group on the local algebras, singling out the local observables as invariants under this action. The vacuum vector is cyclic and separating for the local field algebras. Further, the local commutation relations (governed by an Abelian representation of the braid group) can be formulated in terms of twisted locality (90), as in the familiar case of Fermions [13].²¹ The field algebra \mathcal{F}_a for Anyons [38, 48] is constructed as follows. Due to our Assumption A3, the map

$$\gamma^q \mapsto q, \quad q \in \mathbb{Z}_N \text{ or } \mathbb{Z}$$

establishes an isomorphism of the groups Σ and \mathbb{Z}_N or \mathbb{Z} , respectively. We shall identify Σ with \mathbb{Z}_N or \mathbb{Z} by this isomorphism, and write q instead of γ^q , \mathcal{H}_q and E_q instead of \mathcal{H}_{γ^q} and E_{γ^q} , and so on. We consider the same Hilbert space \mathcal{H} and representation U of the universal covering group of the Poincaré group as before, see Eq.s (13) and (14). The dual $\hat{\Sigma}$ of the group Σ is called the gauge group and is represented on \mathcal{H} via

$$V(t) := \sum_{q \in \Sigma} \exp(2\pi i q t) E_q. \quad (79)$$

The anyonic field algebra \mathcal{F}_a is now the C^* -algebra generated by operators $F_a(c, A)$, $c \in \Sigma$, $A \in \mathcal{A}$, acting as

$$F_a(c, A) : (q, \psi) \mapsto (q + c, \pi_q(A)\psi).$$

Here we have written $\pi_q := \pi_0 \circ \gamma^q$ to save on notation. Clearly there holds $F_a(c_1, A_1)F_a(c_2, A_2) = F_a(c_1 + c_2, \pi_{c_2}(A_1)A_2)$, and therefore \mathcal{F}_a coincides with the closed linear span of the operators $F_a(c, A)$, $A \in \mathcal{A}$, $c \in \Sigma$. Furthermore, the adjoint is given by

$$F_a(c, A)^* = F_a(-c, \gamma^{-c}(A^*)). \quad (80)$$

As in the case of the reduced field bundle, we call a field operator $F_a(c, A)$ localized in $\tilde{I} \in \tilde{\mathcal{K}}$ if there is a charge transporter U for γ^c along \tilde{I} such that $UA \in \mathcal{A}(I)$. The von Neumann algebra generated by these operators is denoted by $\mathcal{F}_a(\tilde{I})$. The adjoint action of the gauge group (79) leaves each local field algebra $\mathcal{F}_a(\tilde{I})$ invariant, the fixed point

²¹Some results on the anyonic field algebra in $d = 2 + 1$ are spread out over the literature, or have not been made very explicit in the accessible literature (see however [38]), namely: Local commutation relations under consideration of the relative winding numbers, twisted duality, and the Reeh-Schlieder property. We therefore collect them, with proofs, in the appendix for the convenience of the reader.

algebra being the direct sum of all $\pi_q(\mathcal{A}(I))$, hence isomorphic to $\mathcal{A}(I)$. The space-like commutation relations have the following explicit form. Let $F_1 = F_a(c_1, A_1) \in \mathcal{F}_a(\tilde{I}_1)$ and $F_2 = F_a(c_2, A_2) \in \mathcal{F}_a(\tilde{I}_2)$, where \tilde{I}_1, \tilde{I}_2 are causally separated and have relative winding number $N(\tilde{I}_2, \tilde{I}_1) = n$. Then there holds²²

$$F_2 F_1 = \omega^{c_1 \cdot c_2 (2n+1)} F_1 F_2,$$

where $\omega \equiv \omega_\gamma$ denotes the statistics phase of the generating automorphism γ , see Eq. (26). This may be reformulated in the form of twisted locality, which by Haag duality (8) sharpens to twisted duality, as follows. For \tilde{I}_1, \tilde{I}_2 causally separated with relative winding number $N(\tilde{I}_2, \tilde{I}_1) = n$, let $Z(\tilde{I}_2, \tilde{I}_1)$ be the unitary operator in \mathcal{H} defined by

$$Z(\tilde{I}_2, \tilde{I}_1) E_q := (\omega^{\frac{1}{2}})^{q^2(2n+1)} E_q, \quad (81)$$

where the root of ω may be chosen at will. (This ‘‘twist operator’’ has been first proposed in [53].)

Lemma 6.1 (Twisted Haag Duality.) *Let \tilde{I}, \tilde{I}' be (classes of) paths in $\tilde{\mathcal{K}}$ ending at I and its causal complement I' , respectively. Then there holds*

$$Z(\tilde{I}, \tilde{I}') \mathcal{F}_a(\tilde{I}') Z(\tilde{I}, \tilde{I})^* = \mathcal{F}_a(\tilde{I}). \quad (82)$$

(We give a proof of this lemma in the appendix.)

6.2 Modular Covariance and CPT Theorem.

We now show that the anyonic field algebra satisfies covariance of the modular groups and conjugations. Since the vacuum is cyclic and separating for the local algebras, see Lemma A.3, the Tomita operator associated with $\mathcal{F}_a(\tilde{W}_1)$ is well-defined. Let us denote this operator and its polar decomposition by

$$S_a = J_a \Delta_a^{1/2}.$$

Theorem 6.2 (Modular Covariance.) *Let the Assumptions A1, A2 and A3 hold. Then the modular unitary group of the anyonic field algebra satisfies modular covariance, namely coincides with the representers of the 1-boosts:*

$$\Delta_a^{it} = U(\tilde{\lambda}_1(-2\pi t)).$$

Proof. The theorem is a simple consequence of our Theorem 4.2 and the following lemma. \square

Lemma 6.3 *The anyonic field algebra \mathcal{F}_a is a sub-algebra of the reduced field bundle \mathcal{F} , and the pseudo-Tomita operator S associated with $\mathcal{F}(\tilde{W}_1)$ and Ω coincides with the Tomita operator S_a associated with $\mathcal{F}_a(\tilde{W}_1)$ and Ω .*

²²See appendix.

(The fact that then S must be an involution, $S^2 \subset \mathbb{1}$, is no contradiction to Eq. (36), since if γ is self-conjugate our Assumption A3 implies that γ is real and not pseudo-real [47, Remark 2 after Lem. 4.5], hence $\chi_\gamma = 1$.)

Proof. Let us first set up the reduced field bundle in the special case at hand. We identify Σ with \mathbb{Z} or \mathbb{Z}_N as before, and denote elements by q, s, c, r . A superselection channel $e = (s, c, r)$ has non-zero intertwiner $T_e \in \text{Int}(s + c|r)$ only if $r = s + c$. In this case we choose $T_e = \mathbb{1}$ and write $F(s, c; A)$ instead of $F(e, A)$. The field algebra \mathcal{F} (alias reduced field bundle) is thus generated by the operators

$$F(s, c; A) : (q, \psi) \mapsto \delta_{s,c} (q + c, \pi_q(A)\psi), \quad (83)$$

$s, c \in \Sigma, A \in \mathcal{A}$. Clearly, $F_a(c, A) = \sum_{s \in \Sigma} F(s, c; A)$, and hence \mathcal{F}_a is a sub-algebra of \mathcal{F} . Further, the map²³ $\mu : \mathcal{F} \mapsto \mathcal{F}_a$ defined by

$$\mu(F(s, c; A)) := F_a(c, A) \quad (84)$$

preserves localization, and for $F \in \mathcal{F}_a$ satisfies $\mu(F)\Omega = F\Omega$ and $\mu(F^\dagger) = \mu(F)^*$. (The last equation follows from

$$F(0, c; A)^\dagger = F(0, -c; \gamma^{-c}(A^*))$$

and Eq. (80).) These relations imply that for $F \in \mathcal{F}_a$ there holds

$$S\mu(F)\Omega = S F\Omega = F^\dagger\Omega = \mu(F^\dagger)\Omega = \mu(F)^*\Omega.$$

But μ clearly maps $\mathcal{F}(\tilde{W}_1)$ onto $\mathcal{F}_a(\tilde{W}_1)$, and the proof is complete. \square

The lemma also implies, of course, that the modular conjugation J_a associated with $\mathcal{F}_a(\tilde{W}_1)$ represents the reflection j in the sense of Eq. (4) or (76). In order to achieve a geometrically correct action on the family of algebras $\mathcal{F}_a(\tilde{I})$ in the sense of Eq. (77), however, one has to multiply it with the twist operator Z ,

$$Z := Z(\tilde{W}_1, j \cdot \tilde{W}_1). \quad (85)$$

Theorem 6.4 (CPT Theorem for Anyons.) *The anti-unitary operator $\Theta := Z^* J_a$ is a CPT operator: It satisfies*

$$\Theta^2 = \mathbb{1}, \quad \Theta U(\tilde{g})\Theta^* = U(j\tilde{g}j), \quad (86)$$

$\tilde{g} \in \tilde{P}_+^\dagger$, and acts geometrically correctly: For all $\tilde{I} \in \tilde{\mathcal{K}}$, there holds

$$\text{Ad}\Theta : \mathcal{F}_a(\tilde{I}) \rightarrow \mathcal{F}_a(j \cdot \tilde{I}). \quad (87)$$

Proof. The commutation relations Eq. (86) of Θ are inherited from those of $J_a \equiv J$, since Z commutes with $U(\tilde{P}_+^\dagger)$. $J_a^2 = \mathbb{1}$, anti-linearity of J_a and $J_a E_q = E_{-q} J_a$ imply that $\Theta^2 = \mathbb{1}$. Tomita-Takesaki's theorem and twisted duality (82) imply that

$$J_a \mathcal{F}_a(\tilde{W}_1) J_a^* = \mathcal{F}_a(\tilde{W}_1)' = Z \mathcal{F}_a(j \cdot \tilde{W}_1) Z^*,$$

²³In fact, μ is a conditional expectation from \mathcal{F} onto \mathcal{F}_a .

which yields Eq. (87) for the case of \tilde{W}_1 . Covariance then implies the geometric action for every $\tilde{I} \in \tilde{\mathcal{K}}$ which ends at a wedge region W , namely, every \tilde{I} of the form $\tilde{g} \cdot \tilde{W}_1$. If \tilde{I} ends at a space-like cone, note that twisted Haag duality (82) implies that \mathcal{F}_a is self-dual, namely

$$\mathcal{F}_a(\tilde{I}) = \bigcap_{\tilde{W} \supset \tilde{I}} \mathcal{F}_a(\tilde{W}),$$

where the intersection goes over all $\tilde{W} \in \tilde{\mathcal{K}}$ which contain \tilde{I} . (By $\tilde{I} \subset \tilde{W}$ we mean that $I \subset W$ and in addition $\tilde{I}^H \subset \tilde{W}^H$ as subsets of \tilde{H} .) Hence Eq. (87) also holds for such \tilde{I} . If \tilde{I} ends at the causal complement of a space-like cone, the equation also holds, since $\mathcal{F}_a(\tilde{I})$ is generated by all $\mathcal{F}_a(\tilde{C})$ with $\tilde{C} \subset \tilde{I}$. (This is so because the analogous statement holds for $\mathcal{A}(I)$.) This completes the proof of the theorem. \square

Up to here, the reference cone C_0 has not been specified, and \tilde{W}_1 may be any path from C_0 to W_1 . The difference in the choices only shows up in the value of the twist operator Z . Recall that there are two topologically distinct choices for C_0 satisfying $jC_0 = C_0$. With any one of these choices, the natural choice of \tilde{W}_1 is the “shortest” path from C_0 to W_1 , namely

$$\tilde{W}_1 := (C_0, I, W_1)$$

for some $I \in \mathcal{K}$. Specifying now C_0 so as to contain the positive or negative x^2 -axis, respectively, the relative winding number of \tilde{W}_1 and $j \cdot \tilde{W}_1$ is $N(\tilde{W}_1, j \cdot \tilde{W}_1) = -1$ or 0, respectively, hence

$$Z E_q = \omega^{\mp \frac{1}{2}q^2} E_q, \quad (88)$$

respectively.

A The Anyon Field Algebra

We collect some results on the anyonic field algebra which are spread out over the literature, or have not been made very explicit in the literature.

Lemma A.1 (Anyonic Commutation Relations.) *Let \tilde{I}_1, \tilde{I}_2 be causally separated, with relative winding number $N(\tilde{I}_2, \tilde{I}_1) = n$.*

i) For $F_1 = F_a(c_1, A_1) \in \mathcal{F}_a(\tilde{I}_1)$ and $F_2 = F_a(c_2, A_2) \in \mathcal{F}_a(\tilde{I}_2)$ there hold the commutation relations

$$F_2 F_1 = \omega^{c_1 \cdot c_2 (2n+1)} F_1 F_2, \quad (89)$$

where ω denotes the statistics phase of the generating automorphism γ , see Eq. (26).

ii) Equivalent with these relations is twisted locality, namely

$$Z(\tilde{I}_2, \tilde{I}_1) \mathcal{F}_a(\tilde{I}_1) Z(\tilde{I}_2, \tilde{I}_1)^* \subset \mathcal{F}_a(\tilde{I}_2) \quad (90)$$

if I_1 and I_2 are causally separated.

Here, $Z(\tilde{I}_2, \tilde{I}_1)$ is the “twist” operator defined in Eq. (81).

Proof. Ad *i*) The commutation relations (23) satisfied by the *reduced* field operators read as follows in the present context of Anyons. Two fields $F(s_1, c_1; A_1) \in \mathcal{F}(\tilde{I}_1)$ and $F(s_2, c_2; A_2) \in \mathcal{F}(\tilde{I}_2)$, where $s_2 = s_1 + c_1$, which are causally separated satisfy the commutation relations

$$F(s_2, c_2; A_2) F(s_1, c_1; A_1) = R(s_1, c_1, c_2; n) F(\hat{s}_1, c_1; A_1) F(s_1, c_2; A_2), \quad (91)$$

where $\hat{s}_1 = s_1 + c_2$, and where n is the relative winding number $N(\tilde{I}_2, \tilde{I}_1)$. The number $R(s_1, c_1, c_2; n)$ is given by, see Eq. (24),

$$R(s_1, c_1, c_2; n) = \left(\frac{\omega_\alpha \omega_\gamma}{\omega_\beta \omega_\delta} \right)^n \pi_0 \gamma^{s_1} (\varepsilon(\gamma^{c_2}, \gamma^{c_1})) \quad (92)$$

with $\alpha = s_1, \beta = s_2 \equiv s_1 + c_1, \gamma = s_2 + c_2 \equiv s_1 + c_1 + c_2, \delta = \hat{s}_1 \equiv s_1 + c_2$. Now the statistics operator $\varepsilon(\gamma^{c_2}, \gamma^{c_1})$ coincides, in the present situation, with $\varepsilon(\gamma, \gamma)^{c_1 \cdot c_2}$ [23, Eq. (2.3)]. Further, $\varepsilon(\gamma, \gamma)$ coincides with a multiple $\omega \mathbb{1}$ of unity [14], where $\omega \equiv \omega_\gamma$ is the statistics phase of γ . Putting all this into Eq. (92) yields $R(s_1, c_1, c_2; n) = \omega^{c_1 c_2 (2n+1)}$. Using this equality, the commutation relations (91) transfer to the fields $F_a(c_i, A_i) \equiv \sum_s F(s, c_i; A_i)$, proving the claim. Ad *ii*) For F_1, F_2 as in (i) and $Z := Z(\tilde{I}_2, \tilde{I}_1)$ one calculates

$$[F_2, ZF_1Z^*] E_q = \omega^{\frac{1}{2}c_1^2 + qc_1} (F_2 F_1 - \omega^{c_1 c_2} F_1 F_2) E_q.$$

Hence the anyonic commutation relations (89) are equivalent with $[F_2, ZF_1Z^*] = 0$. This implies that ZF_1Z^* also commutes with the von Neumann algebra generated by operators of the form F_2 , and completes the proof. \square

Lemma A.2 (Twisted Haag Duality.) *Let \tilde{I}, \tilde{I}' be (classes of) paths in $\tilde{\mathcal{K}}$ ending at I and its causal complement I' , respectively. Then there holds*

$$Z(\tilde{I}, \tilde{I}') \mathcal{F}_a(\tilde{I}') Z(\tilde{I}, \tilde{I}')^* = \mathcal{F}_a(\tilde{I})'. \quad (93)$$

Proof. This follows from twisted locality and Haag duality of the observables as in the permutation group case [17, Thm. 5.4], the argument being as follows in the present setting. A standard argument [16, Remark 1 after Prop. 2.2] using the Reeh-Schlieder property and the action (79) of the gauge group implies that every operator B in $\mathcal{F}_a(\tilde{I})'$ decomposes, just like a field operator, as the sum of operators $B_q \in \mathcal{F}_a(\tilde{I})'$ carrying fixed charge, i.e. $B_c E_q = E_{q+c} B_c$. (Namely, $B_c = \int_{\tilde{\Sigma}} dt \exp(-2\pi i ct) V(t) B V(t)^*$.) The same holds for $Z^* \mathcal{F}_a(\tilde{I})' Z$, where $Z := Z(\tilde{I}, \tilde{I}')$. Let now $F \in Z^* \mathcal{F}_a(\tilde{I})' Z$ carry charge c , and pick a unitary $\Psi \in \mathcal{F}_a(\tilde{I})'$ of the same charge. Then $B := \Psi^* F$ also is in $Z^* \mathcal{F}_a(\tilde{I})' Z$ by twisted locality, and carries charge zero. Therefore it is in $\mathcal{F}_a(\tilde{I})'$, and acts according to $B(q, \psi) = (q, B_q \psi)$. One concludes that for every $q \in \Sigma$, charge transporter U_q for γ^q along the path \tilde{I} and observable $A \in \mathcal{A}(I)$ there holds

$$B_q \pi_0(U_q^* A) = \pi_0(U_q^* A) B_0.$$

Putting $q = 0$, this implies by Haag duality that $B_0 = \pi_0(\hat{B})$ for some $\hat{B} \in \mathcal{A}(I')$. The same equation (with $A = \mathbb{1}$) then implies that $B_q = \pi_0(U_q^* \hat{B} U_q)$, which coincides with $\pi_q(\hat{B})$. Thus, B coincides with $F_a(0, \hat{B})$ and is therefore in $\mathcal{F}_a(\tilde{I})'$, and the same holds for F . This completes the proof. \square

Lemma A.3 (Reeh-Schlieder Property.) *The vacuum is cyclic and separating for every $\mathcal{F}_a(\tilde{I})$, $\tilde{I} \in \tilde{\mathcal{K}}$.*

Proof. Cyclicity of the vacuum $\Omega = (0, \Omega_0)$ for $\mathcal{F}_a(\tilde{I})$ follows from the cyclicity of Ω_0 for $\mathcal{A}(I)$ and the definition $F_a(c, A)(0, \Omega_0) = (c, \pi_0(A)\Omega_0)$. Now by twisted locality (90), $\mathcal{F}_a(\tilde{I})' \Omega$ contains $Z \mathcal{F}_a(\tilde{I}') \Omega$, with Z unitary, which is dense. Hence the vacuum is cyclic for $\mathcal{F}_a(\tilde{I})'$ and therefore separating for $\mathcal{F}_a(\tilde{I})$. \square

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References

- [1] H. Araki, *Mathematical theory of quantum fields*, Int. Series of Monographs in Physics, no. 101, Oxford University Press, 1999.
- [2] J.J. Bisognano and E.H. Wichmann, *On the duality condition for a Hermitean scalar field*, J. Math. Phys. **16** (1975), 985.
- [3] ———, *On the duality condition for quantum fields*, J. Math. Phys. **17** (1976), 303.
- [4] H.J. Borchers, *The CPT-theorem in two-dimensional theories of local observables*, Commun. Math. Phys. **143** (1992), 315–332.
- [5] ———, *On Poincaré transformations and the modular group of the algebra associated with a wedge*, Lett. Math. Phys. **46** (1998), 295–301.
- [6] H.J. Borchers, D. Buchholz, and B. Schroer, *Polarization-free generators and the S-matrix*, Commun. Math. Phys. **219** (2001), 125–140.
- [7] H.J. Borchers and J. Yngvason, *On the PCT-theorem in the theory of local observables*, Mathematical Physics in Mathematics and Physics (Siena) (R. Longo, ed.), Fields Institute Communications, vol. 30, 2001, pp. 39–64.
- [8] O. Bratteli and D.W. Robinson, *Operator algebras and quantum statistical mechanics 1*, second ed., TMP, Springer, New York, 1987.
- [9] R. Brunetti, D. Guido, and R. Longo, *Modular structure and duality in conformal field theory*, Commun. Math. Phys. **156** (1993), 201–219.
- [10] ———, *Modular localization and Wigner particles*, Rev. Math. Phys. **14** (2002), 759–786.
- [11] D. Buchholz and H. Epstein, *Spin and statistics of quantum topological charges*, Fysica **17** (1985), 329–343.
- [12] D. Buchholz and K. Fredenhagen, *Locality and the structure of particle states*, Commun. Math. Phys. **84** (1982), 1–54.
- [13] S. Doplicher, R. Haag, and J.E. Roberts, *Fields, observables and gauge transformations I*, Commun. Math. Phys. **13** (1969), 1–23.
- [14] ———, *Local observables and particle statistics I*, Commun. Math. Phys. **23** (1971), 199.
- [15] ———, *Local observables and particle statistics II*, Commun. Math. Phys. **35** (1974), 49–85.
- [16] S. Doplicher and J.E. Roberts, *Fields, statistics and non-Abelian gauge groups*, Commun. Math. Phys. **28** (1972), 331–348.

- [17] ———, *Why there is a field algebra with a compact gauge group describing the superselection structure in particle physics*, Commun. Math. Phys. **131** (1990), 51–107.
- [18] K. Fredenhagen, *On the existence of antiparticles*, Commun. Math. Phys. **79** (1981), 141–151.
- [19] ———, *Structure of superselection sectors in low dimensional quantum field theory*, Proceedings (Lake Tahoe City) (L.L. Chau and W. Nahm, eds.), 1989.
- [20] ———, *Generalizations of the theory of superselection sectors*, The algebraic theory of superselection sectors. Introduction and recent results. (D. Kastler, ed.), World Scientific, 1990.
- [21] K. Fredenhagen, M. Gaberdiel, and S.M. Rüger, *Scattering states of plektons (particles with braid group statistics) in 2+1 dimensional field theory*, Commun. Math. Phys. **175** (1996), 319–355.
- [22] K. Fredenhagen, K.-H. Rehren, and B. Schroer, *Superselection sectors with braid group statistics and exchange algebras I: General theory*, Commun. Math. Phys. **125** (1989), 201–226.
- [23] ———, *Superselection sectors with braid group statistics and exchange algebras II: Geometric aspects and conformal covariance*, Rev. Math. Phys. **SI1** (1992), 113–157.
- [24] J. Fröhlich and P.A. Marchetti, *Quantum field theories of vortices and anyons*, Commun. Math. Phys. **121** (1989), 177–223.
- [25] ———, *Spin-statistics theorem and scattering in planar quantum field theories with braid statistics*, Nucl. Phys. B **356** (1991), 533–573.
- [26] J. Fröhlich and T. Kerler, *Quantum groups, quantum categories, and quantum field theory*, Lecture Notes in Mathematics, vol. 1542, Springer, Berlin, 1993.
- [27] J. Fuchs, A. Ganchev, and P. Vecsernyés, *Rational Hopf algebras: Polynomial equations, gauge fixing, and low dimensional examples*, Int. J. Mod. Phys. **A 10** (1995), 3431–3476.
- [28] D. Guido and R. Longo, *Relativistic invariance and charge conjugation in quantum field theory*, Commun. Math. Phys. **148** (1992), 521–551.
- [29] ———, *An algebraic spin and statistics theorem*, Commun. Math. Phys. **172** (1995), 517.
- [30] ———, *Natural energy bounds in quantum thermodynamics*, Commun. Math. Phys. **218** (2001), 513–536.
- [31] R. Haag, *Local quantum physics*, second ed., Texts and Monographs in Physics, Springer, Berlin, Heidelberg, 1996.
- [32] K. Hepp, *On the connection between Wightman and LSZ quantum field theory*, Axiomatic Field Theory (M. Chretien and S. Deser, eds.), Brandeis University Summer Institute in Theoretical Physics 1965, vol. 1, Gordon and Breach, 1966, pp. 135–246.

- [33] R. Jost, *The general theory of quantized fields*, American Mathematical Society, Providence, Rhode Island, 1965.
- [34] B. Kuckert, *A new approach to spin & statistics*, Lett. Math. Phys. **35** (1995), 319–331.
- [35] ———, *Two uniqueness results on the Unruh effect and on PCT-symmetry*, Commun. Math. Phys. **221** (2001), 77–100.
- [36] R. Longo, *An analogue of the Kac-Wakimoto formula and black hole conditional entropy*, Commun. Math. Phys. **186** (1997), 451–479.
- [37] G. Mack and V. Schomerus, *Conformal field algebras with quantum symmetry from the theory of superselection sectors*, Commun. Math. Phys. **134** (1990), 139–196.
- [38] J. Mund, *Quantum Field Theory of Particles with Braid Group Statistics in 2+1 Dimensions*, Ph.D. thesis, Freie Universität Berlin, 1998.
- [39] ———, *The Bisognano-Wichmann theorem for massive theories*, Ann. H. Poinc. **2** (2001), 907–926.
- [40] ———, *Modular localization of massive particles with “any” spin in d=2+1*, J. Math. Phys. **44** (2003), 2037–2057.
- [41] ———, *Borchers’ commutation relations for sectors with braid group statistics in low dimensions*, to be published by Ann. H. Poinc., 2009.
- [42] ———, *The spin statistics theorem for anyons and plektons in d=2+1*, Commun. Math. Phys. **286** (2009), 1159–1180.
- [43] J. Mund, B. Schroer, and J. Yngvason, *String-localized quantum fields from Wigner representations*, Phys. Lett. B **596** (2004), 156–162.
- [44] B. O’Neill, *Semi-riemannian geometry*, Academic Press, New York, 1983.
- [45] W. Pauli, *Exclusion principle, Lorentz group and reflection of space-time and charge*, Niels Bohr and the Development of Physics (W. Pauli, ed.), Pergamon Press, 1955, p. 30.
- [46] K.-H. Rehren, *Braid group statistics and their superselection rules*, The Algebraic Theory of Superselection Sectors (D. Kastler, ed.), World Scientific, Singapore, 1990.
- [47] ———, *Spacetime fields and exchange fields*, Commun. Math. Phys. **132** (1990), 461–483.
- [48] ———, *Field operators for anyons and plektons*, Commun. Math. Phys. **145** (1992), 123.
- [49] ———, *Weak C* Hopf symmetry*, Group Theoretical Methods in Physics (A. Bohm, H.-D. Doebner, and P. Kielanowski, eds.), Lecture Notes in Physics, vol. 504, Heron Press, Berlin, 1997, q-alg/9611007, pp. 62–69.
- [50] J.E. Roberts, *Local cohomology and superselection structure*, Commun. Math. Phys. **51** (1976), 107–119.

- [51] ———, *Net cohomology and its applications to field theory*, Quantum Fields – Algebras, Processes (L. Streit, ed.), Springer, Wien, New York, 1980, pp. 239–268.
- [52] ———, *Lectures on algebraic quantum field theory*, The Algebraic Theory of Superselection Sectors. Introduction and Recent Results (D. Kastler, ed.), World Scientific, Singapore, New Jersey, London, Hong Kong, 1990, pp. 1–112.
- [53] B. Schroer, *Modular theory and symmetry in QFT*, Mathematical Physics towards the 21st Century (R.N. Sen and A. Gersten, eds.), Ben-Gurion of the Negev Press, Beer-Sheva, Israel, 1994.
- [54] B. Schroer and H.-W. Wiesbrock, *Modular theory and geometry*, Rev. Math. Phys. **12** (2000), 139–158.
- [55] O. Steinmann, *A Jost-Schroer theorem for string fields*, Commun. Math. Phys. **87** (1982), 259–264.
- [56] S. Strătilă, *Modular theory in operator algebras*, Abacus Press, Tunbridge Wells, England, 1981.
- [57] W.G. Unruh, *Notes on black hole evaporation*, Rev. Math. Phys. **14** (1976), 870–892.
- [58] V.S. Varadarajan, *Geometry of quantum theory*, vol. II, Van Nostrand Reinhold Co., New York, 1970.
- [59] F. Wilczek, *Quantum mechanics of fractional-spin particles*, Phys. Rev. Lett. **49** (1982), 957–1149.